Problem 1:

We can think of X_n as $\left\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \,\middle|\, \begin{matrix} x_0^2+\dots+x_n^2=1, \text{ or} \\ x_0=0 \text{ and } x_1^2+\dots+x_n^2\le 1\end{matrix}\right\}$.

If $U = X_n\cap\{x_n>1/4\}$, $V=X_n\cap\{x_n<3/4\}$, then U, V are contractible, and $X_n\cap\{x_n=1/2\}\subset U\cap V$ is a strong deformation retract. However,

$$X_n\cap\{x_n=1/2\}\cong\left\{(x_0,\dots,x_{n-1})\,\middle|\,\begin{matrix} x_0^2+\dots+x_{n-1}^2=3/4 \text{ or} \\ x_0=0 \text{ and } x_0^2+\dots+x_{n-1}^2\le 3/4\end{matrix}\right\}$$

which is homeomorphic to X_{n-1}. The Mayer Vietoris sequence tells us that $\tilde{H}_{i+1}(X_n)\cong\tilde{H}_i(X_{n-1})$ $\forall i,n$. Since $X_0=\{pt\}\cup\{pt\}\cup\{pt\}$,

$$\tilde{H}_i(X_n)=\begin{cases}\mathbb{Z}^2 & \text{if } i=n\\ 0 & \text{otherwise}\end{cases}.$$

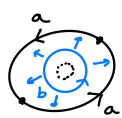
Problem 2:

Thinking of $\mathbb{RP}^2$ as the disk $\overline{D(0,1)}/\sim$, where $\sim$ identifies antipodal points in the boundary, and let $U=D(0,3/4)$, $V=\mathbb{RP}^2\setminus\overline{D(0,1/4)}$ then U,V are open, they cover $\mathbb{RP}^2$, U is contractible, $V\sim S^1$ and $U\cap V\sim S^1$. The reduced M.V sequence reads:

$$0\longrightarrow H_2(\mathbb{RP}^2)\longrightarrow H_1(U\cap V)\xrightarrow{\phi} H_1(V)\longrightarrow H_1(\mathbb{RP}^2)\longrightarrow 0$$

where ϕ maps the generator of $H_1(U\cap V)$ to $2\cdot$the generator of $H_1(V)$.*

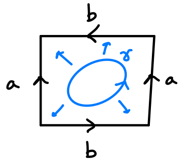
so $H_2(\mathbb{RP}^2)\cong\ker\phi=0$, $H_1(\mathbb{RP}^2)\cong\operatorname{coker}\phi=\mathbb{Z}/2$, $H_0(\mathbb{RP}^2)=\mathbb{Z}$ and the rest of the groups vanish.

* 

visualization of why $\phi(b)=2a$

Problem 3:

Similarly to Problem 2, let $U = D(0,3/4)$, $V = K \setminus D(0,1/4)$. This time,



$V \sim S^1 \vee S^1$, $U \cap V \sim S^1$ and U is contractible. The MV sequence is the same, but this time,

$$\phi : H_1(U \cap V) \longrightarrow H_1(V) = \mathbb{Z}a + \mathbb{Z}b$$

sends γ to $b + a + b - a$, so $H_2(K) \cong \ker(\phi) = 0$ $H_1(K) \cong \operatorname{coker} \phi = \mathbb{Z} \oplus \mathbb{Z}/2$, $H_0(K) = \mathbb{Z}$ and the rest of the groups vanish

Problem 4:

Let U, V be as in Problem 3. Now we have $U \cap V \sim S^1$, $V \sim \bigvee_{i=1}^{2g} S^1$, and U is contractible. The map

$$\phi : H_1(U \cap V) \longrightarrow H_1(V) \cong \bigoplus_{i=1}^{g} \mathbb{Z}a_i \oplus \mathbb{Z}b_i$$

sends γ to $a_1 + b_1 - a_1 - b_1 + \ldots + a_g + b_g - a_g - b_g = 0$. Therefore,

$$H_i(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5

Let's start by trying to approach the homology groups of $M \# \Sigma_1$, where M is any surface.

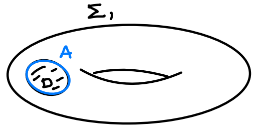
If A is the disk in $M \# \Sigma_1$ that we use to glue the components, it is not hard to see that $(A, M \# \Sigma_1)$ is a good pair, and $M \# \Sigma_1 / A \cong M \vee \Sigma_1$. Using Problem 7, we find exact sequences

(A) $\overset{0}{\overset{\shortparallel}{\tilde{H}_i(A)}} \longrightarrow \tilde{H}_i(M\#\Sigma_1) \longrightarrow \tilde{H}_i(M)\oplus\tilde{H}_i(\Sigma_1) \longrightarrow \overset{0}{\overset{\shortparallel}{\tilde{H}_{i-1}(A)}}$ for $i\geq 3$

and

(B) $0\to \tilde{H}_2(M\#\Sigma_1)\to \tilde{H}_2(M)\oplus\tilde{H}_2(\Sigma_1)\xrightarrow{\beta}\tilde{H}_1(A)\xrightarrow{\gamma}\tilde{H}_1(M\#\Sigma_1)\xrightarrow{\varepsilon}\tilde{H}_1(M)\oplus\tilde{H}_1(\Sigma_1)\to 0$

<u>Claim:</u> The composition $\tilde{H}_2(\Sigma_1)\xrightarrow{(0,id)}\tilde{H}_2(M)\oplus\tilde{H}_2(\Sigma_1)\xrightarrow{\beta}\tilde{H}_1(A)$ is the map α in the LES of $(\Sigma_1\setminus \mathrm{int}(D), A)$, under the identification $\frac{\Sigma_1\setminus\mathrm{int}(D)}{A}\simeq\Sigma_1$.



($\star$) $\tilde{H}_2(\Sigma_1\setminus\mathrm{int}(D))\longrightarrow\tilde{H}_2(\Sigma_1)\xrightarrow{\alpha}\tilde{H}_1(A)\longrightarrow\tilde{H}_1(\Sigma_1\setminus\mathrm{int}(D))$

In particular, β is surjective.

Proof:
$$\begin{tikzcd}
(\Sigma_1\setminus\mathrm{int}(D), A) \arrow[r, "i"] \arrow[d, "q"] & (M\#\Sigma_1, A) \arrow[d, "p"] \\
(\Sigma_1, pt) \arrow[r, "j"] & (M\vee\Sigma_1, pt)
\end{tikzcd}$$
is a commutative diagram of pairs, and we get the commutative diagram, since $i|_A = id$.

$$\begin{tikzcd}
 & \tilde{H}_2(\Sigma_1\setminus\mathrm{int}(D), A) \arrow[r, "\partial_2"] \arrow[d, "q", "\simeq"'] \arrow[dr, "i_*"] & \tilde{H}_1(A) \\
\tilde{H}_2(\Sigma_1) \arrow[r, "\simeq", "\delta"'] & \tilde{H}_2(\Sigma_1, pt) \arrow[dr, "(id{,}0)=j_*"'] & \tilde{H}_2(M\#\Sigma_1, A) \arrow[u, "\partial_1"'] \arrow[d, "p_*", "\simeq"'] \\
 & & \tilde{H}_2(\Sigma_1, pt)\oplus\tilde{H}_2(M, pt)
\end{tikzcd}$$

where ∂_1, ∂_2 are the connecting homomorphisms of the pairs. From this the first statement follows. For the second one, it is enough to show that α is surjective, but we know all the groups in ($\star$):

$0\longrightarrow\mathbb{Z}\xrightarrow{\alpha}\mathbb{Z}\longrightarrow\mathbb{Z}^2$ is exact, so $\mathrm{coker}(\alpha)\hookrightarrow\mathbb{Z}^2$, but $\mathrm{coker}(\alpha)$ is a finite group, so the only way for such an injection to exist is if $\mathrm{coker}(\alpha)=0$; i.e if α is surjective. $\square$.

Now let $M=\Sigma_{g-1}$, and assume that we know that $H_i(\Sigma_{g-1})=\begin{cases}\mathbb{Z} & \text{if } i=0,2\\ \mathbb{Z}^{2g-2} & \text{if } i=1\\ 0 & \text{otherwise}\end{cases}$

the sequences (A) on top of the page show that $H_i(\Sigma_{g-1}\#\Sigma_1)=0$ if $i\geq 3$.

If we look at sequence (B), we see that $\beta : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is surjective, and by exactness we obtain that

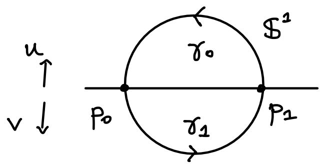
- $\tilde{H}_2(\Sigma_{g-1} \# \Sigma_1) \cong \ker(\beta) = \mathbb{Z}$
- $\mathrm{coker}(\beta) = 0 \Rightarrow \ker(\gamma) = H_1(A) \Rightarrow \varepsilon$ is injective $\Rightarrow \varepsilon$ is an isomorphism, so

$$\tilde{H}_1(\Sigma_{g-1} \# \Sigma_1) \simeq \tilde{H}_1(\Sigma_{g-1}) \oplus \tilde{H}_1(\Sigma_1) = \mathbb{Z}^{2g}$$

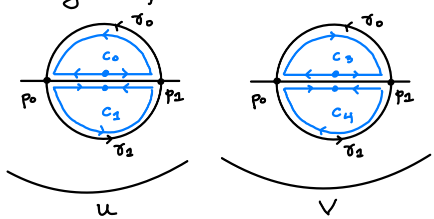
Finally, $H_0(\Sigma_{g-1} \# \Sigma_1) = \mathbb{Z}$ because it is connected.

Problem 6.

A generator of the 0-th reduced homology of $S^0 = \cdot_{p_0} \cdot_{p_1}$ is $[p_0] - [p_1]$.
The boundary map $\tilde{H}_1(S^1) \longrightarrow \tilde{H}_0(S^0)$ sends the cycle $[\gamma_0] + [\gamma_1]$ to $\partial \gamma_0 = [p_0] - [p_1]$, so $[\gamma_0] + [\gamma_1]$ is a generator too:



The boundary map sends $[c_0] + [c_1] + [c_2] + [c_3]$ to $\partial(c_0) + \partial(c_1) = [\gamma_0] + [\gamma_1]$:



Problem 7:

If $W_1 = X \vee_p V$, $W_2 = U \vee_p Y$, W_1, W_2 are open sets in $X \vee Y$, and the homotopies $V \sim p$, $U \sim p$ imply that $W_1 \sim X$, $W_2 \sim Y$ and $W_1 \cap W_2 = U \vee_p V \sim p$.
Reduced MV then reads: $\underset{0}{\underset{\shortparallel}{\tilde{H}_i(W_1 \cap W_2)}} \longrightarrow \tilde{H}_i(X) \oplus \tilde{H}_i(Y) \longrightarrow \tilde{H}_i(X \vee Y) \longrightarrow \underset{0}{\underset{\shortparallel}{\tilde{H}_i(W_1 \cap W_2)}}$