Problem 1

It is enough to show that τ_1 and τ_2 are homotopic, but

$$H_\theta(x_1, x_2, x_3, \dots, x_{n+1}) = (\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2, x_3, \dots, x_{n+1})$$

is a continuous map on $[-\pi/2, \pi/2] \times S^n$

Problem 2

Let X, Y be Hausdorff, locally compact spaces an $f: X \longrightarrow Y$ continuous. then the extended map $\hat{f}: \hat{X} \longrightarrow \hat{Y}$ is continuous if and only if f is proper (f^{-1}(compact) is compact). Rmk: this proves the exercise because any homomorphism is proper

Proof: $\hat{f}$ is continuous at every point of $\hat{X} \setminus \{\infty\}$ because f is. Recall that the open neighbourhoods of ∞ are sets of the form $\{\infty\} \cup (X \setminus K)$, where K is compact in X. Therefore, $\hat{f}$ is continuous at ∞ if and only if $f^{-1}(Y \setminus K) \cup \{\infty\} = Y \setminus f^{-1}(K) \cup \{\infty\}$ is an open neighbourhood of ∞ for any $K \subseteq Y$ compact; i.e. iff f is proper.

So for example, $(0,1) \subset \mathbb{R}$ does not extend to the compactification.

Problem 3:

The stereographic projection is the map

$$S^n \setminus \{(1,0,\dots,0)\} \subseteq \mathbb{R}^{n+1} \xrightarrow{\pi} \mathbb{R}^n$$

$$(x_0, \dots, x_n) \longrightarrow \left(\frac{x_1}{1-x_0}, \dots, \frac{x_n}{1-x_0}\right)$$

with inverse

$$\pi^{-1}(y_1, \dots, y_n) = \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y_1}{\|y\|^2+1}, \dots, \frac{2y_n}{\|y\|^2+1}\right)$$

And so it extends to the compactifications

$$\mathbb{S}^n \cong \widehat{\mathbb{S}^n \setminus \{(1,0,\dots,0)\}} \xrightarrow{\hat{\pi}} \hat{\mathbb{R}}^n$$

Problem 4:

As suggested by the hint, the map

$$\begin{array}{ccc} \mathbb{S}^{2k-1} \times (-\varepsilon,\varepsilon) & \xrightarrow{\phi} & \mathbb{S}^{2k-1} \\ \cap & & \cap \\ \mathbb{C}^k \times (-\varepsilon,\varepsilon) & & \mathbb{C}^k \end{array}$$

given by $\phi(z,t) = e^{it} z$ is a 1-parameter system of neighbourhoods, and the associated vector field is $X(z) = \frac{d}{dt}\phi(z,t)\Big|_{t=0} = i \cdot z$.
In real coordinates, $X(x_1,y_1,\dots,x_k,y_k) = (-y_1,x_1,\dots,-y_k,x_k)$.

Problem 5:

a) Let $g = (-id)\circ f$. Then $g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ satisfies $x + g(x) \neq 0$ for all x. If

$$H(t,x) = \frac{tx + (1-t)\cdot g(x)}{\|tx + (1-t)\cdot g(x)\|}$$

then H is well-defined because $x + g(x) \neq 0$, so $id = H(1,\cdot) \sim H(0,\cdot) = g$

So we have shown that $(-id)\circ f \sim id \Rightarrow f \sim (-id) \Rightarrow \deg(f) = (-1)^{n+1}$

b) Using a) in its contrapositive form,

- $\deg(f) = 0 \neq (-1)^{n+1} \Rightarrow \exists x : f(x) = x$.
- $\deg(-f) = 0 \neq (-1)^{n+1} \Rightarrow \exists y : -f(y) = y$.

Problem 6

generator of $H_1(S^1)$.

Recall that, for a map $\gamma:(S^1,1)\to(X,x_0)$, $\phi([\gamma])=\gamma_*(1)$, where $\phi:\pi_1(X,x_0)\longrightarrow H_1(X)$ is the Hurewicz homomorphism.

On the other hand, if $p:(Y,y_0)\longrightarrow(X,x_0)$ is the universal covering, there is a bijection

$$\pi_1(X,x_0)\xrightarrow{\Psi} p^{-1}(x_0)$$

where $\Psi([\gamma])=\tilde{\gamma}(1)$, where $\tilde{\gamma}:([0,1],0)\longrightarrow(Y,y_0)$ fits into a diagram

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\tilde{\gamma}"] \arrow[d] & Y \arrow[d, "p"] \\
S^1 \arrow[r, "\gamma"] & X
\end{tikzcd}$$

In our situation, $(X,x_0)=(S^1,1)$ and $(Y,y_0)=(\mathbb{R},0)$, with $p(x)=e^{2\pi i x}$, and $\gamma(z)=z^k$.

In this case, $\tilde{\gamma}$ can be explicitly computed:

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\times k"] \arrow[d] & \mathbb{R} \arrow[d, "e^{2\pi i x}"] \\
S^1 \arrow[r, "z\mapsto z^k"] & S^1
\end{tikzcd}$$

and so, $\gamma_*(1)=\tilde{\gamma}(1)=k$, so $\deg(\gamma)=k$.

Alternatively, $z\longrightarrow z^k$ is a smooth map, 1 is a regular value and the preimage of 1 has cardinality $|k|$, and that at each p mapping to 1, $\varepsilon_p=\operatorname{sign}(k)$

Problem 7

a) As we proven in another exercise, there is a commutative diagram

$$\begin{tikzcd}
H_{n+1}(\Sigma S^n) \arrow[r, "\phi"] \arrow[d, "\Sigma f_*"] & H_n(S^n) \arrow[d, "f_*"] \\
H_{n+1}(\Sigma S^n) \arrow[r, "\phi"] & H_n(S^n)
\end{tikzcd}$$

if x is a generator of $H_n(\mathbb{S}^n)$, $\phi^{-1}(x)$ is a generator of $H_{n+1}(\Sigma \mathbb{S}^n)$, and

$$\Sigma f_*(\phi^{-1}(x)) = \phi^{-1} \circ f_* \circ \phi(\phi^{-1}(x)) = \deg(f) \cdot \phi^{-1}(x)$$

and so Σf_* is "multiplication by $\deg(f)$".

b) It follows from part a) and Problem 6.

Problem 8

For $n=1$, let $\gamma: \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the identity path and let σ be the concatenation of γ and γ^{-1}. As discussed in Problem 6,

$$\sigma_*(1) = \phi([\gamma \circ \gamma^{-1}]) = \phi([\gamma]) - \phi([\gamma]) = 0$$

so σ has degree 0.

Alternatively, one can directly define

$$H_t : [0,1] \longrightarrow [0,1]$$
$$s \longmapsto \begin{cases} 2st & \text{if } s \leq 1/2 \\ t(2-2s) & \text{if } s \geq 1/2 \end{cases}$$

and H_t descends to a homotopy between $H_0 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ (the constant map) and $H_1 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ (a surjective map), so $\deg(H_1) = 0$.

For higher n just use Problem 7, a) and that, if $f: X \longrightarrow Y$ is surjective, so is $\Sigma f : \Sigma X \longrightarrow \Sigma Y$.

Problem 9

$GL(n, \mathbb{C})$ is connected: Given $A \in GL(n, \mathbb{C})$, let $z \in \mathbb{C}^*$ be such that the line $\mathbb{R} \cdot z$ does not contain any of the eigenvalues of A. Then

$$\gamma(t) = t \cdot I + (1-t) \cdot z^{-1} \cdot A \quad , t \in [0,1]$$

is a path contained in $GL(n, \mathbb{C})$ because for $t < 1$,

$$\det(\gamma(t)) = \left(\frac{1-t}{z}\right)^n \cdot \det\left(A + z \frac{t}{1-t} I\right)$$

which is non-zero because $z \frac{t}{t-1}$ is not an eigenvalue of A.

$SO(n)$ is connected: $SO(2)$ is homeomorphic to $\mathbb{S}^1$. Then argue by induction. If x is any point on the sphere and $\gamma: [0,1] \longrightarrow \mathbb{S}^{n-1}$ is a path between $(1,0,\dots,0)$ and x, the Gram-Schmidt algorithm allows one to obtain a path $\tilde{\gamma}: [0,1] \longrightarrow SO(n)$ with $\tilde{\gamma}(0) = Id$ and $\tilde{\gamma}(t)(1,0,\dots,0) = \gamma(t)$. Therefore, we can connect any matrix in $SO(n)$ to one where the first column is $(1,0,\dots,0)^t$, but this forces the first row to be $(1,0,\dots,0)$ and so the matrix has the form $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)$, with $A \in SO(n-1)$, and we can use induction to reach the identity.

$GL(n,\mathbb{R})^+$ is connected: for $n=1$ this is clear. If $n>1$, any point in $\mathbb{R}^n \setminus \{0\}$ can be connected by a path to $(1,0,\dots,0)$. Therefore, there is a path in $GL(n,\mathbb{R})^+$ connecting any matrix to one of the form $\left(\begin{array}{c|c} 1 & v \\ \hline 0 & B \end{array}\right)$, where now $B \in GL(n-1,\mathbb{R})^+$. Doing the same construction for B and so on, we see that there is a path in $GL(n,\mathbb{R})^+$ between any matrix and one of the form $\begin{pmatrix} 1 & & \nabla \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, but $\left\{A \in GL(n,\mathbb{R})^+ : A = \begin{pmatrix} 1 & & \nabla \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$ is homeomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$, which is connected.