Solutions AT-I Winter 16

1. (a) Let $i : A \hookrightarrow X$ be the inclusion and $r : X \to A$ the retraction. Since $r \circ i = id_A$ the induced maps on homology satisfy $r_* \circ i_* = (id_A)_*$ and therefore i_* must be injective.

   (b) Since i_* is injective, the long exact sequence of the pair (X, A) splits up into short exact sequences of the form:

   $$0 \to H_j(A) \xrightarrow{i_*} H_j(X) \to H_j(X, A) \to 0.$$

   The induced map r_* is a left inverse of i_* and therefore, by the "splitting Lemma", we know that this short exact sequence *splits* and hence we have $H_j(A) \oplus H_j(X, A) \cong H_j(X)$.

   (c) No. For any map $g : X \to Y$, the mapping cylinder always deformation retracts onto Y (see exercise sheet 1), i.e. $M_g \simeq Y$. Hence, M_f deformation retracts onto the S^1 corresponding to the image of f. Since the degree of f is 2 we have:

   $$\begin{tikzcd} & H_1(S^1) \\ H_1(S^1) \arrow[ur, "\cdot 2"] \arrow[r, "i_*"'] & H_1(M_f) \arrow[u, "\cong"'] \end{tikzcd}$$

   Assuming that there is a retraction as in the question, by parts a) and b), we get a split short exact sequence

   $$0 \to H_1(S^1) \to H_1(M_f) \to H_1(M_f, S^1) \to 0$$

   and hence

   $$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

   But $\mathbb{Z} \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}$ and hence the sequence does not split. This is a contradiction to part b) and therefore there is no retract from M_f to the $S^1 \subset M_f$ corresponding to the domain of f.

2. (a) Suppose $\exists x \in S^n$ which is not covered by $c = \sum_j n_j \sigma_j$. That is $x \notin \bigcup_j image(\sigma_j)$, where $\sigma_j : \Delta^n \to S^n$. Then each σ_j can be factored as

   $$\begin{tikzcd} \Delta^n \arrow[r, "\sigma_j'"] \arrow[rr, bend right, "\sigma_j"'] & S^n \setminus \{x\} \arrow[r, hook, "i"] & S^n. \end{tikzcd}$$

   Let $c' := \sum_j n_j \sigma_j' \in S_n(S^n \setminus \{x\})$, as c is a cycle, c' is a cycle as well and $[c'] \in H_n(S^n \setminus \{x\}) \cong H_n(\mathbb{R}) = 0$. Hence $[c'] = 0$ and therefore we get that $[c] = i_*[c'] = 0$ and the claim is established.

(b)
- $\underline{n=1}$: As $\deg f|_{S^{n-1}} \neq 0$, the map $f|_{S^{n-1}} : S^0 \to S^0$ is surjective. The image $f(D^1)$ is connected and hence f is surjective.
- $\underline{n \geq 2}$: Assume that f is not surjective, then there is a point $p \in \operatorname{int}(D^n) \setminus \operatorname{image}(f)$. We get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\dots \arrow[r] & H_n(D^n) \arrow[r] \arrow[d] & H_n(D^n, S^{n-1}) \arrow[r] \arrow[d] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d] & H_{n-1}(D^n) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(D^n \setminus \{p\}) \arrow[r] & H_n(D^n \setminus \{p\}, S^{n-1}) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(D^n \setminus \{p\}) \arrow[r] & \dots
\end{tikzcd}$$

We have that $H_n(D^n) = H_{n-1}(D^n) = 0$, $H_n(D^n \setminus \{p\}) \cong H_n(S^{n-1}) = 0$ and $H_n(D^n \setminus \{p\}, S^{n-1}) \cong H_n(S^{n-1}, S^{n-1}) = 0$ and hence we obtain:

$$\begin{tikzcd}
H_n(D^n, S^{n-1}) \arrow[r, "\cong"] \arrow[d] & H_{n-1}(S^{n-1}) \arrow[d, "f_*"'] \\
0 \arrow[r] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

By commutativity of the diagram, the isomorphism in the top row composed with f_* must be zero. But $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is just multiplication by $\deg f|_{S^{n-1}}$. So we get a contradiction to the assumption that $\deg f|_{S^{n-1}} \neq 0$.

3. (a)
- Take any point on the equator of $\partial B^3 = S^2$ as the unique 0-cell x.
- x and the points equivalent to it divide the equator into p segments; choose any one of them as the unique 1-cell a.
- Any point on the upper hemisphere of $\partial B^3 = S^2$ gets identified with exactly one other point. Take the upper hemisphere as the unique 2-cell b.
- Take B^3 as the unique 3-cell c. The attaching map is given by the quotient map.

(b)
- We clearly have $\mathrm{d}(x) = 0$.
- Since there is only one 0-cell we have $d(a) = x - x = 0$.
- $d(b) = \pm p \cdot a$. (The rotation by $2\pi/p$ of the equator is orientation preserving).
- $d(c) = \pm(b - b) = 0$ since the reflection along the equator of S^2 is orientation reversing.

Therefore the cellular homology of L_p is given by:

$$H_n(L_p) \cong \begin{cases} \mathbb{Z}, & n = 3 \\ 0, & n = 2 \\ \mathbb{Z}_p, & n = 1 \\ \mathbb{Z}, & n = 0. \end{cases}$$

4. (a) Let $\pi : S^{2k} \to \mathbb{R}P^{2k}$ be the usual $2:1$-covering. Since S^{2k} is simply connected (as $k > 1$) the map $\pi \circ f : S^{2k} \to \mathbb{R}P^{2k}$ can be lifted to a map $\tilde{f} : S^{2k} \to S^{2k}$.

$$\begin{tikzcd} S^{2k} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & S^{2k} \arrow[d, "\pi"] \\ \mathbb{R}P^{2k} \arrow[r, "f"] & \mathbb{R}P^{2k} \end{tikzcd}$$

   By a Corollary of the lecture there exists $x \in S^{2k}$ such that $\tilde{f}(x) = \pm x$. Since $\pi(x) = \pi(-x)$ we get $f \circ \pi(x) = \pi \tilde{f}(x) = \pi x$ and therefore $\pi(x)$ is a fixed point of f.

   (b) Fix any point $p \in S^k \subset \mathbb{R}^{n+1}$. If $k = 0$, the constant map with image p has degree 0 and the fixed point p.

   If $k \geq 1$, choose k disjoint closed balls $B_j \subset S^n$ which don't contain p. Define the map $g_k : S^n \to S^n$ as follows: g_k maps the interior of each of the balls B_j homeomorphically onto $S^n \setminus \{p\}$ and the boundary of ∂B_j to $p \in S^n$. The complement $S^n \setminus \bigcup B_j$ is mapped to p by g_k. Then g_k has degree k and $g_k(p) = p$.

   If $k < 0$, compose the previous map with a reflection along a hypersurface in $\mathbb{R}^{n+1}$ that contains p.

5. (a) No. Consider for example the quotient map $p : S^n \to \mathbb{R}P^n$ where $n \geq 2$. Any singular simplex of $\mathbb{R}P^n$ can be lifted to a singular simplex of S^n and therefore the induced map on the singular chains $\pi_* : S_\bullet(S^n) \to S_\bullet(\mathbb{R}P^n)$ is surjective. However, the induced map on the homology level cannot be surjective because $H_k(S^n) = 0$ and $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ for any odd k with $1 \leq k < n$.

   (b) No. Consider for example an embedding of S^1 into the disk D^2. Since it is an embedding, the induced map on the singular chains is injective. However, the induced map on the homology level is not injective since $H_1(S^1) \cong \mathbb{Z}$ and $H_1(D^2) = 0$.

   (c) Yes. Let $\psi = \varphi^{-1} : D_\bullet \to C_\bullet$. Then ψ is a chain map

$$\partial \psi = \psi \circ \varphi \circ \partial \psi = \psi \circ \partial \varphi \circ \psi = \psi \partial.$$

On the homology level we have $\psi_*\varphi_* = (\psi\varphi)_* = (id_C)_* = id_{H(C)}$ and $\varphi_*\psi_* = (\varphi\psi)_* = (id_D)_* = id_{H(D)}$. Therefore φ_* is an isomorphism on homology with inverse ψ_*.

6. (a) The differential on the cellular chain complex is given by: $d\sigma = \sum_\tau [\tau : \sigma]\tau$, where τ ranges of all k-cells of X. Since $d^2 = 0$ we get:

$$d^2\sigma = \sum_\tau [\tau : \sigma] d\tau = \sum_{\tau,\kappa} [\tau : \sigma][\kappa : \tau]\kappa = 0,$$

where κ ranges over all $(k-1)$-cells of X. If ω is any $(k-1)$-cell of X, the coefficient of $d^2\sigma$ in front of ω must vanish and this coefficient is precisely given by $\sum_\tau [\tau : \sigma][\omega : \tau]$. This proves the claim.

(b) The cellular map g induces a chain map g_{CW} from the cellular chain complex of X to the cellular chain complex of Y, i.e. $d \circ g_{CW} = g_{CW} \circ d$. We have

$$g_{CW}(d\eta) = \sum_\lambda [\lambda : \eta] g_{CW}(\lambda) = \sum_{\xi,\lambda} [\lambda : \eta] \deg(g_{\xi,\lambda})\xi \tag{1}$$

$$d \circ g_{CW}(\eta) = \sum_\alpha \deg(g_{\alpha,\eta}) d\alpha = \sum_{\alpha,\xi} \deg(g_{\alpha,\eta})[\xi : \alpha]\xi \tag{2}$$

where λ ranges over all $(k-1)$-cells of X, α ranges over all k-cells of Y and ξ ranges over all $(k-1)$-cells of Y. Since (1)=(2), the coefficients in front of the $(k-1)$-cell β of Y in both equations must be equal. Therefore we get:

$$\sum_\lambda [\lambda : \eta] \deg(g_{\beta,\lambda}) = \sum_\alpha \deg(g_{\alpha,\eta})[\beta : \alpha].$$