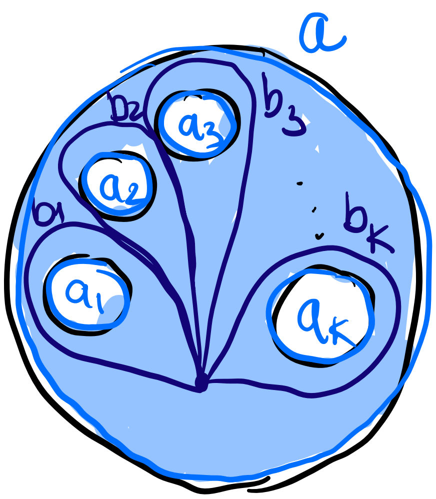
Additional Exercises, Sheet 3, #5

Let D be a 2-disc with k open discs removed. Compute the homology groups of the pair $(D, \partial D)$.



∂D: disjoint union of $(k+1)$-circles S^1:

$$\tilde{H}_i(\partial D) = \begin{cases} \mathbb{Z}^k & i=0 \\ \mathbb{Z}^{k+1} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

D is homotopy equivalent to a wedge of k circles:

$$\tilde{H}_i(D) = \begin{cases} \mathbb{Z}^k & i=1 \\ 0 & \text{otherwise} \end{cases}$$

↪ prove using Δ-complexes (you may assume singular homology groups are isomorphic to simplicial homology groups)

↳ It is also true that:

For a wedge sum $\bigvee_\alpha X_\alpha$ the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ s.t. the pairs (X_α, x_α) are good, ie. x_α is a strong def. retract of a neighborhood of x_α V in X.

(Hatcher, page 126 - coming soon)

LES

$$\dots \to \tilde{H}_n(\partial D) \to \tilde{H}_n(D) \to H_n(D, \partial D) \to \tilde{H}_{n-1}(\partial D) \to \dots$$

When $i \geq 3$

$$\dots \underbrace{\tilde{H}_i(\partial D)}_{0} \to \underbrace{\tilde{H}_i(D)}_{0} \to H_i(D, \partial D) \to \underbrace{\tilde{H}_{i-1}(\partial D)}_{0} \to \underbrace{\tilde{H}_{i-1}(D)}_{0} \to \dots$$

and by exactness $H_i(D, \partial D) \cong \tilde{H}_{i-1}(\partial D) = 0$.

Observe

$$\dots \to \underbrace{\tilde{H}_2(\partial D)}_{0} \to \underbrace{\tilde{H}_2(D)}_{0} \to H_2(D, \partial D) \xrightarrow[\partial_*^2]{} \underbrace{\tilde{H}_1(\partial D)}_{\langle a_1, a_2, \dots, a_k, a \rangle} \xrightarrow{i_*^1}$$

$$\to \tilde{H}_1(D) \xrightarrow{j_*^2} H_1(D,\partial D) \xrightarrow{\partial_*^1} \tilde{H}_0(\partial D) \to \cancel{\tilde{H}_0(D)} \to H_0(D,\partial D) \to 0$$

$\langle b_1, b_2, \ldots, b_k \rangle$ $\qquad$ $\mathbb{Z}^k$ $\qquad$ 0

The map $i_*^1 : \tilde{H}_1(\partial D) \to \tilde{H}_1(D)$

$$a_1 \longmapsto b_1$$
$$\vdots$$
$$a_k \longmapsto b_k$$
$$a \longmapsto \sum_{i=1}^{k} b_i$$

i_*^1 is a surjective map & the kernel is generated by $a - \sum_{i=1}^{k} a_i$.

∂_*^2 is injective (exactness at $H_2(D,\partial D)$)

and so

$$H_2(D,\partial D) \cong \mathrm{Im}\,\partial_*^2 = \ker i_*^1 \cong \mathbb{Z}$$

Since i_*^1 is surjective, $\mathrm{Im}\, i_*^1 = \tilde{H}_1(D) \cong \ker j_*^2$ & so j_*^2 is the zero map. Because of this ∂_*^1 is injective. Since $\tilde{H}_0(D)$

is trivial, ∂^1_* is also surjective & therefore an isomorphism.

$$\Rightarrow H_1(D, \partial D) \cong \tilde{H}_0(\partial D) \cong \mathbb{Z}^k.$$

Finally, $H_0(D, \partial D) = 0$

($0 \to H_0(D, \partial D) \to 0 \to 0$ is exact).

So

$$H_i(D, \partial D) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^k & i = 1 \\ 0 & \text{otherwise} \end{cases}$$