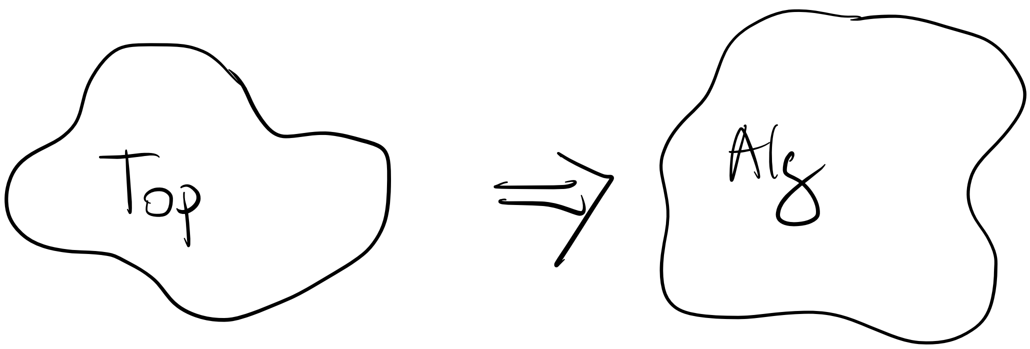
INTRODUCTION

What is topology?

It is a study of topological spaces up to a homeomorphism (or some other equivalence)

One way to study topological spaces is by using ALGEBRA $\rightsquigarrow$ ALGEBRAIC TOPOLOGY

How: by assigning algebraic objects to topological spaces



$X \longmapsto G(X)$ Invariants of topological spaces

If $X \cong Y \Rightarrow G(X) \cong G(Y)$

Examples:

- FUNDAMENTAL GROUP $\pi_1(X, x_0)$
  (point-set topology class at ETH)

- HIGHER HOMOTOPY GROUPS

CONVENTIONS:

- 'space' = 'topological space'
- X topological space, $A \subset X$ with the induced topology (from X) is called a subspace.
- $f: X \to Y$ 'map' = 'continuous map'
- $A \subset X$, $B \subset Y$, $f: (X, A) \to (Y, B)$ means such a map $f: X \to Y$ s.t. $f(A) \subset B$.

QUOTIENT TOPOLOGY

Let X be a topological space, Y a set, $g: X \to Y$ surjective (onto).

Define a topology on Y as follows:

$V \subset Y$ open $\iff g^{-1}(V) \subset X$ is open.

This is the finest topology that makes g continuous. It is called the **QUOTIENT TOPOLOGY** on Y.

Examples:

1) X topological space, $\sim$ an equivalence relation on X. Let $Y = X/_\sim$ (the set of all equivalence classes). Then

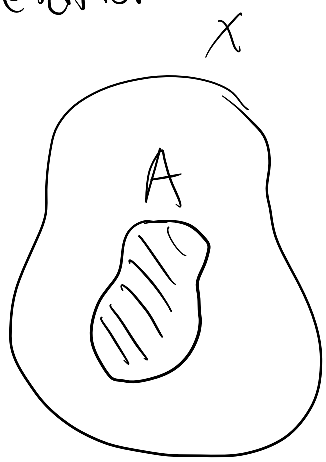
$$g: X \to Y \text{ is surjective}$$
$$g(x) = [x]$$

and we can equip Y with the quotient

topology.

2) X = topological space, $A \subset X$ subspace.

We can define an equivalence relation on X as follows:



$x \sim y$ if either $x, y \in A$
$x = y$

The equivalence classes are:

$$\{[x]\}_{x \in X \setminus A}, [A]$$

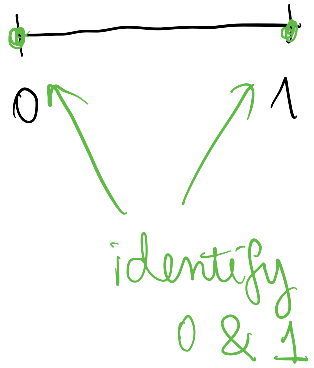
The quotient space X/A is equipped with the quotient topology.

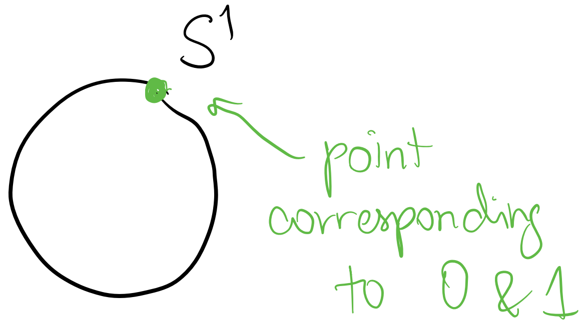
WARNING: This definition is not the same as the group theory definition of G/H, where G is a group & H is a subgroup.

Example: (1) $I = [0,1]$, $A = \partial I = \{0,1\}$

Then $\quad I/A \cong S^1$ circle

homeo.





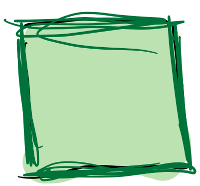
More generally, $\quad I^n/\partial I^n \cong S^n.$ n-dimensional sphere

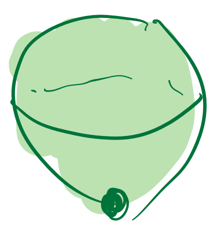
Here $\quad I^n = \underbrace{I \times I \times I \cdots \times I}_{n \text{ times}} = \{(x_1, \dots, x_n) \mid x_i \in I\}$

$$\partial I^n = \{(x_1, \dots, x_n) \in I^n \mid \exists j \text{ s.t. } x_j = 0 \text{ or } x_j = 1\}$$

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

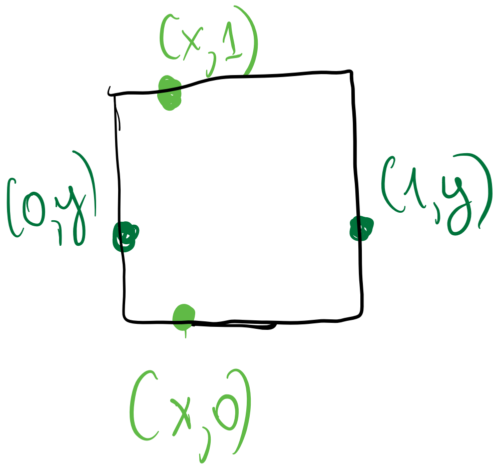
$n=2$





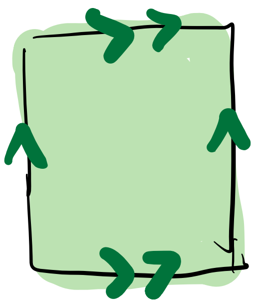
<u>Exercise</u>.

(2) $X = I \times I$



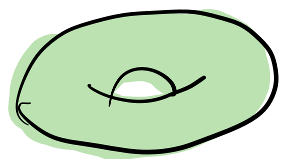
$(x,0) \sim (x,1) \quad \forall x \in I$

$(0,y) \sim (1,y) \quad \forall y \in I$



$I^2/\sim \; \cong T^2$

2-dimensional torus (or donut)



HOMOTOPY

DEFINITION

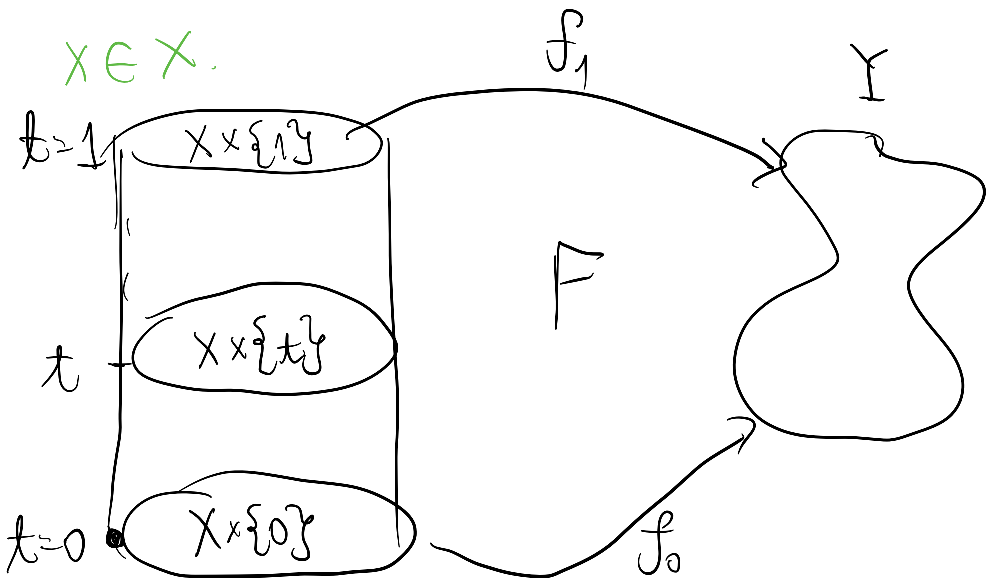
Let X, Y be topological spaces. A **HOMOTOPY** of maps from X to Y is a map

$$F : X \times I \to Y.$$

Equivalently, F is a continuous 1-parameter family of maps $f_t : X \to Y$, where $f_t(x) = F(x,t)$, $t \in I$.

DEFINITION

Two maps f_0 and $f_1 : X \to Y$ are said to be **HOMOTOPIC** if there exists a homotopy $F : X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$.



Notation

We write $f_0 \simeq f_1$ if f_0 is homotopic to f_1.

Example Any two maps $f, g : X \to \mathbb{R}^2$ are homotopic. Homotopy (called LINEAR HOMOTOPY) is given by

$$x \longmapsto (1-t) f(x) + t g(x), \quad x \in X, t \in I.$$

Proposition

$$\begin{tikzcd} X \arrow[r, "f", shift left] \arrow[r, "g"', shift right] & Y \arrow[d, "k"] \\ X' \arrow[u, "h"] & Y' \end{tikzcd}$$

If $f \simeq g$, then $f \circ h \simeq g \circ h$ and $k \circ f \simeq k \circ g$.

Exercise.

DEFINITION

A map $f : X \to Y$ is called a **HOMOTOPY EQUIVALENCE** if $g : Y \to X$ exists s.t.

$$g \circ f \simeq id_X \quad \text{and} \quad f \circ g \simeq id_Y.$$

When such f & g exist, the spaces X & Y are said to be **HOMOTOPY EQUIVALENT** or have the same **HOMOTOPY TYPE**.

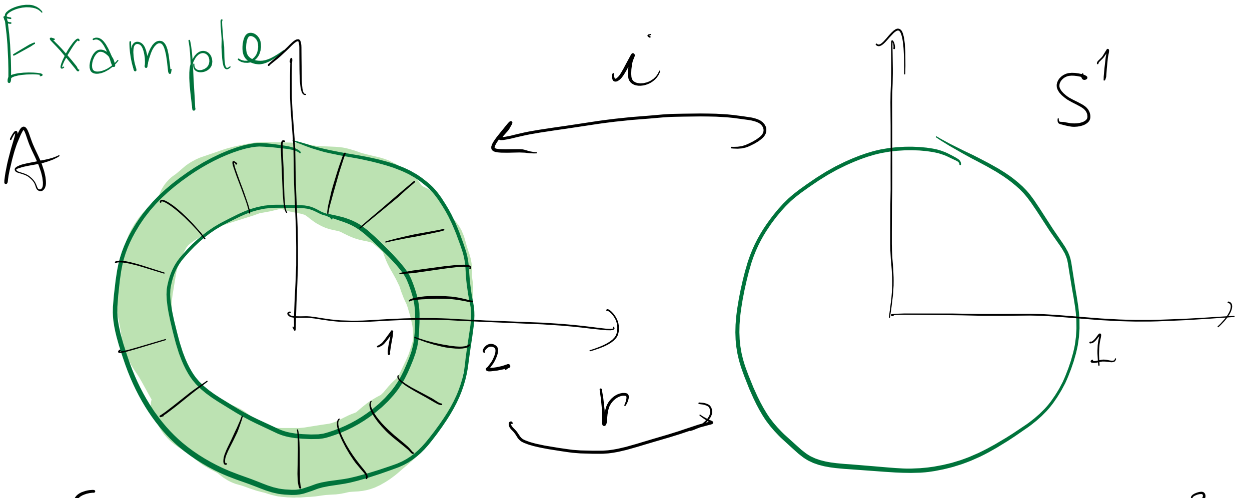
Notation: $X \simeq Y$.

PROPOSITION

$\simeq$ is an equivalence relation on topological spaces.

Exercise.

Example



$\{(x,y) \mid 1 \le x^2+y^2 \le 4\}$ $\qquad$ $\{(x,y) \mid x^2+y^2=1\}$

$i : S^1 \hookrightarrow A \qquad (x,y) \mapsto (x,y)$

$r : A \to S^1 \qquad (x,y) \mapsto \dfrac{(x,y)}{\sqrt{x^2+y^2}}$

$r \circ i : S^1 \to S^1$ is the identity map.

$i \circ r : A \to A$

$F : A \times I \to A$

$$F((x,y),t) = t(x,y) + (1-t)\frac{(x,y)}{\sqrt{x^2+y^2}}$$

This map is continuous

$$F((x,y),0) = \frac{(x,y)}{\sqrt{x^2+y^2}} = r(x,y)$$

$$F((x,y),1) = (x,y) = id_A(x,y)$$

So $F : A \times I \to A$ is a homotopy between $i \circ r$ and id_A.

Therefore, annulus and circle are homotopy equivalent.

They are not homeomorphic.
(thoughts on how to prove it?)

THERE EXIST HOMOTOPY EQUIVALENT SPACES THAT ARE NOT HOMEOMORPHIC.

DEFINITION

A space X is called **CONTRACTIBLE** if X is homotopy equivalent to the one-point space.

$$X \underset{i}{\overset{c}{\rightleftarrows}} \{x_0\} \qquad c \circ i = id, \quad i \circ c \simeq id$$

PROPOSITION

Let X be a space, $x_0 \in X$. Let $c : X \to X$ be the constant map $c(x) = x_0 \ \forall x \in X$.

X is contractible $\iff c \simeq id_X$.

Proof

(X is contractible $\Leftarrow c \simeq id_X$)

Let $c : X \to X$ be such that $c(x) = x_0$.

Let $i : \{x_0\} \to X$

$r : X \to \{x_0\}$

Then $r \circ i = id_{\{x_0\}}$ & $\underset{\overset{\shortparallel}{c}}{i \circ r} \underset{F}{\simeq} id_X$.

$\Rightarrow X$ is contractible. $\quad$ $\uparrow$ by assumption

(X is contractible $\Rightarrow$ $c \simeq id_X$)

$c(x) = x_0$ for $x \in X$.

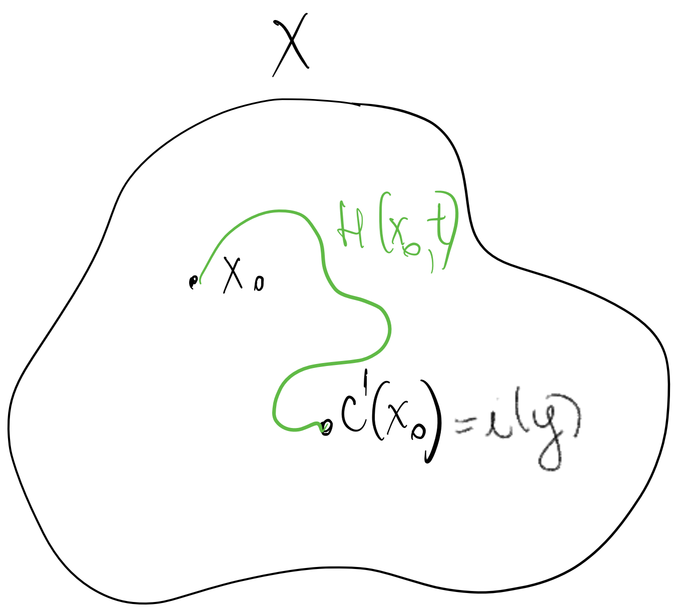
Since X is contractible, there exist i & r

$i : \{y\} \to X$ such that
$r : X \to \{y\}$

$r \circ i = id_{\{y\}}$

$\underbrace{i \circ r}_{\substack{\text{constant} \\ \text{map } c'}} \underset{H}{\simeq} id_X$

$x \mapsto i(y)$

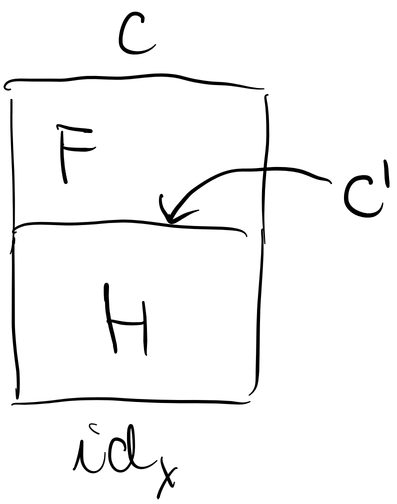


$H(x_0, 0) = x_0$
$H(x_0, 1) = c'(x_0)$

$H(x_0, t)$ is a path from x_0 to $c'(x_0) = i(y)$

Homotopy between c' and c is given by

$$F(x,t) = H(x_0, 1-t) \text{ for } t \in [0,1].$$



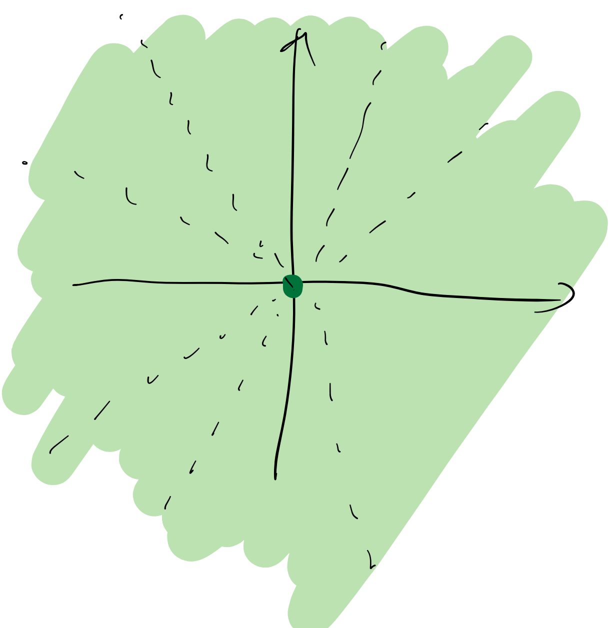
Homotopy between id_x & C is given by

$$H * F = \begin{cases} H(x,2t) & 0 \le t \le \frac{1}{2} \\ F(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

concatenation of homotopies

Example

$X = \mathbb{R}^n$ is contractible.



this is a homotopy between id & a constant map

$$F(x,t) = t \cdot x$$

$$F(x,0) = 0$$

$$F(x,1) = x = id_{\mathbb{R}^n}(x)$$