we can prove the following theorem [For this part we follow Hatcher]

THEOREM

If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \to H_n(Y)$$

In particular, if f is a homotopy equivalence, then f_* is an isomorphism for all n.

Proof

The essential ingredient of the proof is to subdivide $\Delta^n \times I$ into simplices.

For a general n:

Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$

and $\Delta^1 \times \{1\} = [w_0, \dots, w_1]$,

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$.

We pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n-simplices each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0.

First step: $[v_0,\dots,v_n] \to [v_0,\dots,v_{n-1},w_n]$

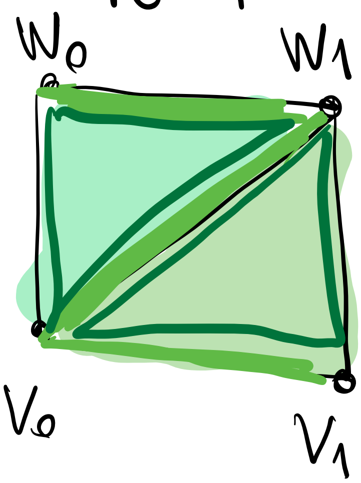
Second step: $[v_0,\dots,v_{n-1},w_n] \to [v_0,\dots,w_{n-1},w_n]$

$\vdots$

$$[v_0,\dots,v_i,w_{i+1},\dots,w_n] \to [v_0,\dots,v_{i-1},w_i,\dots,w_n]$$

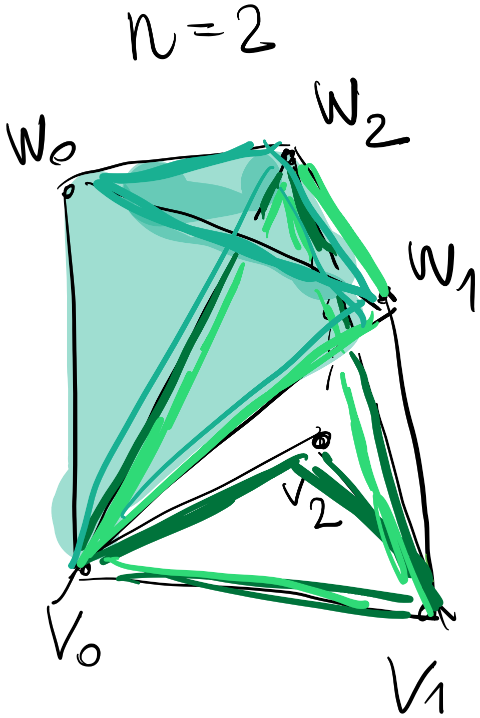
The region between these two simplices is exactly the $(n+1)$-sx $[v_0,\dots,v_i,w_i,\dots,w_n]$ which has $[v_0,\dots,v_i,w_{i+1},\dots,w_n]$ as a lower face and $[v_0,\dots,v_{i-1},w_i,\dots,w_n]$ as an upper face.

$n=1$



$$\left.\begin{array}{l}[v_0,v_1] \to \\ [v_0,w_1] \to \\ [w_0,w_1]\end{array}\right\}\ \text{sequence of 1-simplices}$$

Regions in between that are 2-simplices: $[v_0, v_1, w_1]$, $[v_0, w_0, w_1]$



$[v_0, v_1, v_2] \to$

$[v_0, v_1, w_2] \to$

$[v_0, w_1, w_2] \to$

$[w_0, w_1, w_2]$

Regions in between that are 3-simplices: $[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Altogether, $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an n-simplex face.

Given a homotopy $F: X \times I \to Y$ from f to g we define

$$P_n : S_n(X) \to S_{n+1}(Y),$$

a homomorphism of groups given on generators by the following formula:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times \mathrm{id}_I)\Big|_{[v_0, \dots, v_j, w_j, \dots, w_n]}}_{\text{these are singular } (n+1)\text{-simplices}}$$

this operator is called the

PRISM OPERATOR.

We will show that prism operators satisfy the basic relation

$$\partial P = g_c - f_c - P\partial$$

top, bottom & the sides of the prism

geometrically ∂P represents the boundary of the prism

To prove this relation we calculate

$$\partial P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times id_I)\Big|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id_I)\Big|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

The terms with $i=j$ in the two sums cancel except for

$F\circ(\sigma\times id_I)|_{[\hat{v}_0, w_0, \ldots, w_n]}$, which

is $g\circ\sigma = g_c(\sigma)$, and $-F\circ(\sigma\times id_I)|_{[v_0,\ldots,v_n,\hat{w}_n]}$

which is $-f\circ\sigma = -f_c(\sigma)$.

The terms with $i\neq j$ are exactly $-P\partial(\sigma)$ since

$$P\partial(\sigma) = \sum_{i<j}(-1)^i(-1)^j F\circ(\sigma\times id_I)|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]} +$$

$$\sum_{i>j}(-1)^{i-1}(-1)^j F\circ(\sigma\times id_I)|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}$$

Now we finish the proof of the theorem.

If $c \in S_n(X)$ is a cycle, then

$$g_c(c) - f_c(c) = \partial P(c) + P\partial(c)$$
$$= \partial P(c)$$

Since $\partial c = 0$. This means that $g_c(c) - f_c(c)$ is a boundary and so $[g_c(c)] = [f_c(c)]$. This implies that g_* equals f_* on the homology class of c.

$\blacksquare$