Quick Intro to Homological Algebra

Let us first recall the definition of a chain complex & homology of a general chain complex.

Definition

A **CHAIN COMPLEX** is a sequence of abelian groups C_i, $i \in \mathbb{Z}$, together with a sequence of homomorphisms $\partial_i : C_i \to C_{i-1}$ s.t. $\partial_{i-1} \circ \partial_i = 0 \;\; \forall i$ (sometimes written as $\partial \circ \partial = 0$).

∂ is called the **BOUNDARY OPERATOR**.

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} C_{i-2} \xrightarrow{\partial} \cdots$$

cycles

Let $Z_i = \ker\left(C_i \xrightarrow{\partial} C_{i-1}\right)$ and

$B_i = \mathrm{Im}\left(C_{i+1} \xrightarrow{\partial} C_i\right)$ ← boundaries

Since $\partial \circ \partial = 0$, we have $B_i \subset Z_i$.

Define $H_i(\mathcal{C}_\bullet) := Z_i / B_i$.

↑ homology in degree i

$\mathcal{C}_\bullet = (C_\bullet, \partial_\bullet)$

Definition [morphism of chain complexes]

If $\mathcal{A}_\bullet = (A_\bullet, \partial^A_\bullet)$, $\mathcal{B}_\bullet = (B_\bullet, \partial^B_\bullet)$ are chain complexes, a **CHAIN MAP**

$f : \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ is a

collection of homomorphisms

$f : A_i \to B_i \ \forall i$ s.t. $f \circ \partial^A = \partial^B \circ f$

$$\begin{tikzcd}
\cdots & A_{i+1} \arrow[r, "\partial_A"] \arrow[d, "f"] \arrow[dr, phantom, "©"] & A_i \arrow[r, "\partial_A"] \arrow[d, "f"] \arrow[dr, phantom, "©"] & A_{i-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots & B_{i+1} \arrow[r, "\partial_B"] & B_i \arrow[r, "\partial_B"] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

all squares are commutative

PROPOSITION

Let $f: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a chain map. Then f induces a homomorphism

$$f_*: H_i(\mathcal{A}_\bullet) \to H_i(\mathcal{B}_\bullet)$$

for all $i \in \mathbb{Z}$ by the following procedure:

Let $\alpha \in H_i(\mathcal{A}_\bullet)$. Pick a cycle $a \in A_i$ s.t. $[a] = \alpha$. Define

$$f_*(\alpha) = [f(a)].$$

Moreover, if $\mathcal{A}_\bullet$, $\mathcal{B}_\bullet$ and $\mathcal{C}_\bullet$ are chain complexes and $f: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, $g: \mathcal{B}_\bullet \to \mathcal{C}_\bullet$ are chain maps, then $g \circ f$ is also a chain map and $(g \circ f)_* = g_* \circ f_*$ and $(id_{\mathcal{A}_\bullet})_* = id_{H_i(\mathcal{A}_\bullet)}$ for all i.

Key ingredient: Chain maps map boundaries to boundaries and cycles to cycles.

EXACT SEQUENCES

Let A, B, C be abelian groups, and

$A \xrightarrow{i} B \xrightarrow{j} C$ be two homomorphisms.

The sequence $A \xrightarrow{i} B \xrightarrow{j} C$ is called EXACT if $\ker j = \operatorname{Im} i$.

A sequence $\cdots \to A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} \cdots$ is called EXACT if $A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1}$ is EXACT for all k.

Remark

1. $0 \to A \xrightarrow{f} B$ is exact $\Leftrightarrow$ f is injective ($\ker f = \{0\}$)

② $A \xrightarrow{g} B \to 0$ is exact $\iff$

g is surjective ($\operatorname{Im} g = \ker 0 = B$)

③ $0 \to A \xrightarrow{h} B \to 0$ is exact $\iff$

h is an isomorphism.

④ If $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, the embedding $i : A \hookrightarrow B$ and the surjection j induce an isomorphism $B/i(A) \xrightarrow{\cong} C$

(this holds since j induces an isomorphism $B/\ker j \to \operatorname{Im} j$)

$$\begin{array}{ccc} B/\ker j & \longrightarrow & \operatorname{Im} j \\ \| & & \| \\ B/i(A) & & C \end{array}$$

An exact sequence $0 \to A \to B \to C \to 0$ is called a **SHORT EXACT SEQUENCE (SES)**.

Let $\mathcal{A}_\bullet, \mathcal{B}_\bullet, \mathcal{C}_\bullet$ be chain complexes.
Let $i : \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, $j : \mathcal{B}_\bullet \to \mathcal{C}_\bullet$ be chain maps. We can look at the sequence

$$0 \to \mathcal{A}_\bullet \xrightarrow{i} \mathcal{B}_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0 \qquad (*)$$

We say that this sequence is exact if $\forall n \in \mathbb{Z}$ $\quad 0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$

is exact.

We call $(*)$ a **SES of chain complexes**.

THEOREM

Let $0 \to \mathcal{A}_\bullet \xrightarrow{i} \mathcal{B}_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0$

be a SES of chain complexes. Then

it induces a **LONG EXACT SEQUENCE IN HOMOLOGY**

$$
\begin{aligned}
\cdots \to H_{n+1}(\mathcal{A}_\bullet) &\xrightarrow{i_*} H_{n+1}(\mathcal{B}_\bullet) \xrightarrow{j_*} H_{n+1}(\mathcal{C}_\bullet) \xrightarrow{\partial_*} \\
\to H_n(\mathcal{A}_\bullet) &\xrightarrow{i_*} H_n(\mathcal{B}_\bullet) \xrightarrow{j_*} H_n(\mathcal{C}_\bullet) \xrightarrow{\partial_*} \\
\to H_{n-1}(\mathcal{A}_\bullet) &\xrightarrow{i_*} H_{n-1}(\mathcal{B}_\bullet) \xrightarrow{j_*} H_{n-1}(\mathcal{C}_\bullet) \xrightarrow{\partial_*} \cdots
\end{aligned}
$$

The homomorphism $\partial_* : H_n(\mathcal{C}_\bullet) \to H_{n-1}(\mathcal{A}_\bullet)$

is called the **CONNECTING HOMOMORPHISM**.

Proof

Let's examine what happens on the chain level in degrees p and $p-1$:

$$\begin{tikzcd}
 & & \exists b \arrow[r, bend left, mapsto] \arrow[ddd, bend right, mapsto] & c & \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"'] & B_p \arrow[r, "j"] \arrow[d, "\partial"'] & C_p \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0 \\
 & a \arrow[r, mapsto] & \partial b & &
\end{tikzcd}$$

We will define $\partial_*: H_p(\mathcal{C}_\bullet) \to H_{p-1}(\mathcal{A}_\bullet)$ as follows.

Let $m \in H_p(\mathcal{C}_\bullet)$. Choose a cycle $c \in C_p$

(ie. $\partial c = 0$) s.t. $[c] = m$.

$B_p \xrightarrow{j} C_p$ is a surjection, so

$\exists b \in B_p$ s.t. $j(b) = c$.

Since $j\partial(b) = \partial(j(b)) = \partial c = 0$,

$\partial(b) \in \ker j = \operatorname{Im} i$.

$\Rightarrow \exists!\ a \in A_{p-1}$ s.t. $i(a) = \partial b$.

Note that

$$i(\partial a) = \partial i(a) = \partial(\partial b) = 0.$$

But i is injective, hence $\partial a = 0$,

ie. a is a cycle.

Define $\partial_*(m) := [a]$.

CLAIM: the definition of ∂_* is good, ie. it doesn't depend on the choice of c (with $[c] = m$) nor on the choice of b.