THEOREM

Let $0 \to \mathcal{A}_\bullet \xrightarrow{i} \mathcal{B}_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0$

be a SES of chain complexes. Then

It induces a **LONG EXACT SEQUENCE IN HOMOLOGY**

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(\mathcal{A}_\bullet) \arrow[r, "i_*"] & H_{n+1}(\mathcal{B}_\bullet) \arrow[r, "j_*"] & H_{n+1}(\mathcal{C}_\bullet) \arrow[dll, "\partial_*"'] \\
& H_n(\mathcal{A}_\bullet) \arrow[r, "i_*"] & H_n(\mathcal{B}_\bullet) \arrow[r, "j_*"] & H_n(\mathcal{C}_\bullet) \arrow[dll, "\partial_*"'] \\
& H_{n-1}(\mathcal{A}_\bullet) \arrow[r, "i_*"] & H_{n-1}(\mathcal{B}_\bullet) \arrow[r, "j_*"] & H_{n-1}(\mathcal{C}_\bullet) \arrow[r, "\partial_*"] & \cdots
\end{tikzcd}
$$

The homomorphism $\partial_* : H_n(\mathcal{C}_\bullet) \to H_{n-1}(\mathcal{A}_\bullet)$

is called the **CONNECTING HOMOMORPHISM**.

Proof

Let's examine what happens on the chain level in degrees p and $p-1$:

$$
\begin{tikzcd}
 & & \exists b \arrow[r, mapsto, bend left] & c & \\
 & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"'] & B_p \arrow[r, "j"] \arrow[d, "\partial"'] & C_p \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0 \\
 & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \\
 & a \arrow[r, mapsto] & \partial b & &
\end{tikzcd}
$$

We will define $\partial_* : H_p(\mathcal{C}_\bullet) \to H_{p-1}(\mathcal{A}_\bullet)$ as follows.

Let $m \in H_p(\mathcal{C}_\bullet)$. Choose a cycle $c \in C_p$ (ie. $\partial c = 0$) s.t. $[c] = m$.

$B_p \xrightarrow{j} C_p$ is a surjection, so $\exists b \in B_p$ s.t. $j(b) = c$.

Since $j\partial(b) = \partial(j(b)) = \partial c = 0$,

$\partial(b) \in \ker j = \operatorname{Im} i$.

$\Rightarrow \exists!\ a \in A_{p-1}$ s.t. $i(a) = \partial b$.

Note that

$$i(\partial a) = \partial i(a) = \partial(\partial b) = 0.$$

But i is injective, hence $\partial a = 0$,

ie. a is a cycle.

Define $\partial_*(\gamma) := [a]$.

CLAIM: the definition of ∂_* is good, ie. it doesn't depend on the choice of c (with $[c] = \gamma$) nor on the choice of b.

Proof of claim:

Fix first c and suppose that $c = j(b')$. Define a' as before but using b'.

$j(b-b') = 0$ (since $j(b) = j(b') = c$)

$\Rightarrow b - b' \in \ker j = \operatorname{Im} i$, so

$b - b' = i(a_0)$ for some $a_0 \in A_p$.

$\partial b - \partial b' = \partial i(a_0) = i(\partial a_0)$

But $\partial b - \partial b' = i(a) - i(a') = i(a - a')$.

So $i(a-a') = i(\partial a_0)$. Since i is injective, $a - a' = \partial a_0 \Rightarrow [a] = [a']$.

We'll show next that the definition of $\partial_* m$ is independent of c (with $[c] = m$).

Consider another cycle $c' \in C_p$ with $[c'] = m$.

$\Rightarrow c' = c + \partial c''$.

Since j is surjective, we may choose b with $j(b) = c$, and b'' with $j(b'') = c''$.

Put

$$b' := b + \partial b''$$

$$j(b') = j(b) + j(\partial b'') = c + \partial j(b'') =$$
$$= c + \partial c'' = c'$$

So b' serves as an element that is sent to c' by j. Now consider $\partial b'$, then take the unique $a' \in A_{p-1}$ with $i(a') = \partial b'$. But

$$\partial b' = \partial b + \partial\partial b'' = \partial b.$$

So

$$i(a') = \partial b' = \partial b = i(a).$$

Since i is injective, $a' = a$.

This completes the proof of the claim.

CLAIM: ∂_* is a homomorphism.

Proof of claim:

Let $c^1, c^2 \in C_p$ be two cycles.

From the recipe for ∂_*, we choose $b^1, b^2 \in B_p$, $a^1, a^2 \in A_{p-1}$ with $i(a^1) = \partial b^1$,

$i(a^2) = \partial b^2$.

Then $\partial_*[c^1] = [a^1]$, $\partial_*[c^2] = [a^2]$.

To apply ∂_* on $[c^1] + [c^2] = [c^1 + c^2]$ we can choose $c^1 + c^2$ to be the representative of $[c^1] + [c^2]$.

$j(b^1+b^2)=j(b^1)+j(b^2)=c^1+c^2$ and

$i(a^1+a^2)=i(a^1)+i(a^2)=\partial b^1+\partial b^2=$

$=\partial(b^1+b^2)$,

$$\Rightarrow \partial_*([c^1]+[c^2])=[a^1+a^2]=[a^1]+[a^2]$$

$$=\partial_*[c^1]+\partial_*[c^2].$$

This proves the claim.

PROOF that the long sequence

$$\cdots \to H_{n+1}(\mathcal{A}_\bullet)\xrightarrow{i_*}H_{n+1}(\mathcal{B}_\bullet)\xrightarrow{j_*}H_{n+1}(\mathcal{C}_\bullet)\xrightarrow{\partial_*}$$

$$\to H_n(\mathcal{A}_\bullet)\xrightarrow{i_*}H_n(\mathcal{B}_\bullet)\xrightarrow{j_*}H_n(\mathcal{C}_\bullet)\xrightarrow{\partial_*}$$

$$\to H_{n-1}(\mathcal{A}_\bullet)\xrightarrow{i_*}H_{n-1}(\mathcal{B}_\bullet)\xrightarrow{j_*}H_{n-1}(\mathcal{C}_\bullet)\xrightarrow{\partial_*}\cdots$$

is exact.

We must verify six statements:

$\operatorname{Im} i_* \subseteq \ker j_*$, $\ker j_* \subseteq \operatorname{Im} i_*$,
$\operatorname{Im} j_* \subseteq \ker \partial_*$, $\ker \partial_* \subseteq \operatorname{Im} j_*$,
$\operatorname{Im} \partial_* \subseteq \ker i_*$ and $\ker i_* \subseteq \operatorname{Im} \partial_*$.

① $\operatorname{Im} i_* \subset \ker j_*$

This follows since $ji = 0$ implies that $j_* i_* = 0$.

② $\operatorname{Im} j_* \subset \ker \partial_*$

Let $\beta \in H_p(B_\bullet)$ and let $b \in B_p$ be a cycle with $\beta = [b]$.

$$\begin{tikzcd}
 & & b \arrow[r, mapsto, bend left] \arrow[dddl, bend right, mapsto] & j(b) & \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"] & B_p \arrow[r, "j"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0 \\
 & 0 \arrow[r, mapsto] & 0 & &
\end{tikzcd}$$

By construction $\partial_*(j_*[b]) = 0$

③ $\operatorname{Im} \partial_* \subseteq \ker i_*$

We must show that $i_* \partial_* = 0$.

Assume that $c \in C_p$ is a cycle.

$i_* \partial_* [c] = [i(a)]$, where $a \in A_{p-1}$

is such that $i(a) = \partial b$, where

$j(b) = c$. $\Longrightarrow$

$$i_* \partial_* [c] = [i(a)] = [\partial b] = 0.$$

④ $\ker j_* \subseteq \operatorname{Im} i_*$:

Assume that $j_*[b] = 0$, where b

is a cycle. Since $j_*[b] = [j(b)]$,

it follows that $j(b) = \partial c$ for some

$c \in C_{p+1}$. Pick b' with $j(b') = c$.

Note that

$$j(b - \partial b') = j(b) - j(\partial b') =$$

$$= \partial c - \partial(j(b')) = \partial c - \partial c = 0$$

By exactness of the SES

$\exists a$ s.t.

$$i(a) = b - \partial b'.$$

Let us show that a is a cycle:

$$\underset{\substack{\| \\ i(\partial a)}}{\partial i(a)} = \partial b - \partial\partial b' = 0 - 0 = 0$$

Since i is injective, $\partial a = 0$.

$$\Rightarrow i_*([a]) = [i(a)] = [b - \partial b']$$

$$= [b]$$

$\Rightarrow [b] \in \operatorname{Im}(i_*)$.

⑤ $\ker i_* \subset \operatorname{Im} \partial_*$

Suppose that $i_*[a] = 0 \Rightarrow$

$i(a) = \partial b$ for some $b \in B_p$.

Put $c := j(b)$. We have

$$\partial(c) = \partial j(b) = j\partial(b) = ji(a) = 0.$$

$\Rightarrow c$ is a cycle. Now by the definition of ∂_*, $\partial_*[c] = [a]$.

⑥ $\ker \partial_* \subset \operatorname{Im} j_*$

Suppose $\partial_*[c] = 0$ for some cycle $c \in C_p$. Choose $b \in B_p$ with $j(b) = c$, and $a \in A_{p-1}$ with $i(a) = \partial b$.

Since $\partial_*[c] = [a] = 0$, $a = \partial a'$

for some $a' \in A_p$.

Now: $\partial i(a') = i\partial a' = i(a) = \partial b$

$\Rightarrow \partial(b - i(a')) = 0$

It follows that $b - i(a')$ is a cycle.

We have:

$$j(b - i(a')) = j(b) - \underbrace{j \circ i(a')}_{0}$$

$$= j(b) = c$$

and from here

$$j_*[b - i(a')] = [c].$$

$\blacksquare$

This kind of method of proof is called **DIAGRAM CHASING**.

ADDENDUM to theorem SES $\Rightarrow$ LES

Let

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_\bullet \arrow[r, "i"] \arrow[d, "f"] & \mathcal{B}_\bullet \arrow[r, "j"] \arrow[d, "g"] & \mathcal{C}_\bullet \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \mathcal{A}'_\bullet \arrow[r, "i'"] & \mathcal{B}'_\bullet \arrow[r, "j'"] & \mathcal{C}'_\bullet \arrow[r] & 0 \end{tikzcd}$$

be two SES of chain complexes and f, g, h chain maps s.t. the diagram above commutes. For $\forall p$ we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{I}"] & B_p \arrow[r, "j"] \arrow[d, "g"] \arrow[dr, phantom, "\mathrm{II}"] & \mathcal{C}_p \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A'_p \arrow[r, "i'"'] & B'_p \arrow[r, "j'"'] & \mathcal{C}'_p \arrow[r] & 0 \end{tikzcd}$$

Then we obtain two LES in homology with maps between them that makes all the squares commutative

$$
\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_p(\mathcal{A}_\bullet) \arrow[r, "i_*"] \arrow[d, "f_*"] \arrow[dr, phantom, "1"] & H_p(\mathcal{B}_\bullet) \arrow[r, "j_*"] \arrow[d, "g_*"] \arrow[dr, phantom, "2"] & H_p(\mathcal{C}_\bullet) \arrow[r, "\partial_*"] \arrow[d, "h_*"] \arrow[dr, phantom, "3"] & H_{p-1}(\mathcal{A}_\bullet) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_p(\mathcal{A}'_\bullet) \arrow[r, "i'_*"] & H_p(\mathcal{B}'_\bullet) \arrow[r, "j'_*"] & H_p(\mathcal{C}'_\bullet) \arrow[r, "\partial_*"] & H_{p-1}(\mathcal{A}'_\bullet) \arrow[r] & \cdots
\end{tikzcd}
$$

Proof

1 & 2 commute because I & II on the chain level commute.

To prove commutativity of 3:

Let $c \in \mathcal{C}_p$ be a cycle.

$\partial_*[c] = [a]$, where $a \in A_{p-1}$ with

$i(a) = \partial(b)$, where $j(b) = c$.

Consider $f(a)$. Firstly,

$$
\left.
\begin{aligned}
& i' \circ f(a) = g \circ i(a) = g \circ \partial(b) = \partial\, g(b) \\
& \text{and} \\
& j' \circ g(b) = h \circ j(b) = h(c).
\end{aligned}
\right\} \text{definition of } \partial_*
$$

It follows that

$$\partial'_* \circ h_* [c] = \partial'_* [h(c)] = [f(a)] = f_*[a] = f_* \partial_* [c].$$

$\blacksquare$

THE 5-LEMMA

Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

be a commutative diagram of abelian groups with exact rows.

① If b & d are injective & a is surjective

$\Rightarrow c$ is injective.

② If b & d are surjective & e is injective

$\Rightarrow c$ is surjective.

③ If a, b, d, e are isomorphisms $\Rightarrow$

c is an isomorphism

Definition

Let $f, g : A_\bullet \to B_\bullet$ be chain maps.

A **CHAIN HOMOTOPY** from f to g is a sequence of homomorphisms h_*

$$h_p : A_p \to B_{p+1} \quad , p \in \mathbb{Z}$$

for which

$$\partial_{p+1} \circ h_p + h_{p-1} \circ \partial_p = g_p - f_p$$

$$\begin{tikzcd}
\cdots \arrow[r] & A_{p+1} \arrow[r] \arrow[d, shift right, "g_{p+1}"'] \arrow[d, shift left, "f_{p+1}"] & A_p \arrow[r] \arrow[dl, "h_p"'] \arrow[d, shift right, "g_p"'] \arrow[d, shift left, "f_p"] & A_{p-1} \arrow[r] \arrow[dl, "h_{p-1}"'] \arrow[d, shift right, "g_{p-1}"'] \arrow[d, shift left, "f_{p-1}"] & \cdots \\
\cdots \arrow[r] & B_{p+1} \arrow[r] & B_p \arrow[r] & B_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

Example

Let $f, g : X \to Y$ be continuous maps