Definition

Let $f, g : A_\bullet \to B_\bullet$ be chain maps.

A **CHAIN HOMOTOPY** from f to g is a sequence of homomorphisms h_*

$$h_p : A_p \to B_{p+1}, \quad p \in \mathbb{Z}$$

for which

$$\partial_{p+1} \circ h_p + h_{p-1} \circ \partial_p = g_p - f_p$$

$$\begin{tikzcd}
\cdots \arrow[r] & A_{p+1} \arrow[r] \arrow[d, shift right, "g_{p+1}"'] \arrow[d, shift left, "f_{p+1}"] & A_p \arrow[r] \arrow[dl, "h_p"'] \arrow[d, shift right, "g_p"'] \arrow[d, shift left, "f_p"] & A_{p-1} \arrow[r] \arrow[dl, "h_{p-1}"'] \arrow[d, shift right, "g_{p-1}"'] \arrow[d, shift left, "f_{p-1}"] & \cdots \\
\cdots \arrow[r] & B_{p+1} \arrow[r] & B_p \arrow[r] & B_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

Example

Let $f, g : X \to Y$ be continuous maps

$P : C_k(X) \to C_{k+1}(Y)$ (prism operator)

P is a chain homotopy from f_c to g_c.

A chain map $f: A_\bullet \to B_\bullet$ is a **CHAIN EQUIVALENCE** if there exists a chain map $g: B_\bullet \to A_\bullet$ and chain homotopies from $f \circ g$ to id and from $g \circ f$ to id.

Two chain complexes are chain equivalent if there exists a chain equivalence between them.

Proposition

If $f, g: A_\bullet \to B_\bullet$ are chain homotopic, then $f_* = g_*$.

Proof

Let $f \overset{h}{\simeq} g$.

Let a be a cycle, ie $\partial a = 0$.

Then

$$g(a) - f(a) = \partial(h(a)) + h(\overset{0}{\overset{\shortparallel}{\partial a}})$$

$$= \partial(h(a))$$

$\uparrow$ this is a boundary

$$\Rightarrow [g(a)] = [f(a)] \Rightarrow g_* = f_*.$$