We need two more definitions from homological algebra in order to define relative homology groups.

Definition

Let $\mathcal{D} = (D_\bullet, \partial_\bullet^D)$ be a chain complex.

A chain complex $\mathcal{C} = (C_\bullet, \partial_\bullet^C)$ is a

CHAIN SUBCOMPLEX if

$C_p \subset D_p \ \forall p$ and if $\partial^C = \partial^D|_{C_\bullet}$

Definition

If $\mathcal{C} \subset \mathcal{D}$ is a subcomplex, then we can define the **QUOTIENT COMPLEX**

$\mathcal{D}/\mathcal{C}$:

maps induced on quotients

$$\cdots \to D_{p+1}/C_{p+1} \xrightarrow{\partial} D_p/C_p \xrightarrow{\partial} D_{p-1}/C_{p-1} \to \cdots$$

$\partial \circ \partial = 0$ since the boundary maps are induced by the boundary maps of $\mathcal{D}$.

RELATIVE HOMOLOGY

Let X be a space and $A \subset X$ a subspace. We denote by $S_\bullet(A)$, $S_\bullet(X)$ the chain complexes of singular chains in A, and in X.

$S_\bullet(A) \subset S_\bullet(X)$ is a subcomplex.

Denote by $i: S_\bullet(A) \to S_\bullet(X)$ the inclusion. This is a chain map.

Let

$$S_\bullet(X,A) := S_\bullet(X) / S_\bullet(A),$$

where

$$S_p(X,A) = S_p(X) / S_p(A)$$

group of singular p-chains in A

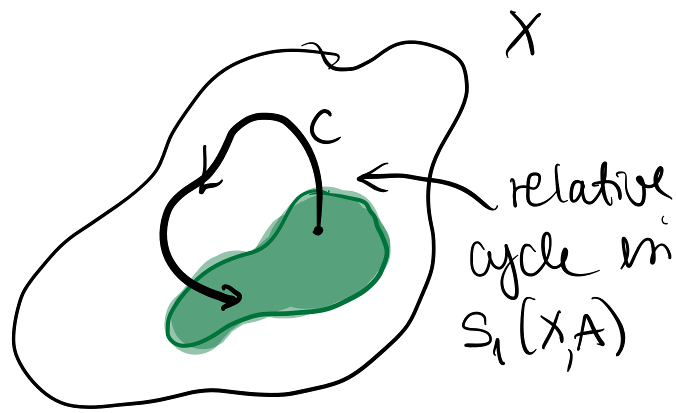
So, $S_\bullet(X,A)$ is

$$\cdots \to S_p(X)/S_p(A) \xrightarrow{\partial} S_{p-1}(X)/S_{p-1}(A) \xrightarrow{\partial} S_{p-2}(X)/S_{p-2}(A) \to \cdots$$

The homology groups of this chain complex are called **RELATIVE HOMOLOGY GROUPS**.

Intuition:

the elements of $H_p(X,A)$ are represented by **RELATIVE CYCLES**, p-chains $c \in S_p(X)$ such that $\partial c \in S_{p-1}(A)$.



A relative cycle c is trivial in $H_p(X,A)$ iff it is a **RELATIVE BOUNDARY**, i.e. if $c = \partial b + a$ for some $b \in S_{p+1}(X)$ and $a \in S_p(A)$.

These properties make precise the intuitive idea that $H_p(X,A)$ is 'homology of X modulo A'.

We have the following SES of chain complexes:

$$0 \to S_\bullet(A) \xrightarrow{i} S_\bullet(X) \xrightarrow{j} S_\bullet(X,A) \to 0.$$

This SES of chain complexes induces a LES in homology

$$\cdots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X,A) \xrightarrow{\partial_*} H_{p-1}(A) \to \cdots$$

The connecting homomorphism has a simple description.

$$\partial_* : H_p(X,A) \to H_{p-1}(A)$$

$$\bar{c} \overset{j}{\longmapsto} c$$

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r, "i"] \arrow[d, "\partial"] & S_p(X) \arrow[r, "j"] \arrow[d, "\partial"] & S_p(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & S_{p-1}(A) \arrow[r, "i"] & S_{p-1}(X) \arrow[r, "j"] & S_{p-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$\bar{c} \in S_p(X)$, $c \in S_p(X,A)$

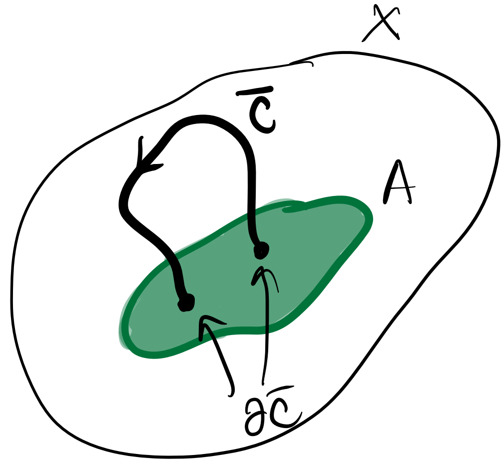
$e \in S_{p-1}(A)$, $\partial\bar{c} \in S_{p-1}(X)$

$$e \longmapsto \partial\bar{c}$$

$\partial\bar{c} \in S_{p-1}(A)$, so we can take $e = \partial\bar{c}$

$$\partial_*[c] = [e] = [\partial \bar{c}]$$

the class of $\partial \bar{c}$ in A



Exactness implies that if $H_p(X, A) = 0$ for all p, then the inclusion $A \hookrightarrow X$ induces isomorphisms $H_p(X) \approx H_p(A)$ $\forall p$.

So we can think of $H_p(X, A)$ as measuring the difference between the groups $H_p(X)$ and $H_p(A)$.

There is an analogous LES of reduced homology groups for a pair (X, A) with $A \neq \phi$. This comes from applying the LES theorem to $0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X,A) \to 0$

in non-negative dimensions,

augmented by the SES $0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0 \to 0$.

in dimension -1.

here we augment by 0, not $\mathbb{Z}$!

Example

LES for (X, x_0), where $x_0 \in X$ yields

$$\cdots \overset{0}{\cancel{\tilde{H}_p(x_0)}} \to \tilde{H}_p(X) \to H_p(X, x_0) \to$$

$$\overset{0}{\cancel{\tilde{H}_{p-1}(x_0)}} \to \tilde{H}_{p-1}(X) \to \cdots$$

$\Rightarrow H_p(X, x_0) \cong \tilde{H}_p(X)$ for all p.

in a few weeks

Soon, we will prove the following theorem:

THEOREM [Hatcher]

If X is a space and A is a non-empty closed subspace that is a strong deformation retract of some neighborhood in X, there is an exact sequence

$$\cdots \to \tilde{H}_p(A) \xrightarrow{i_*} \tilde{H}_p(X) \xrightarrow{j_*} \tilde{H}_p(X/A) \xrightarrow{\partial_*}$$

$$\to \tilde{H}_{p-1}(A) \xrightarrow{i_*} \tilde{H}_{p-1}(X) \xrightarrow{j_*} \tilde{H}_{p-1}(X/A) \to \cdots$$

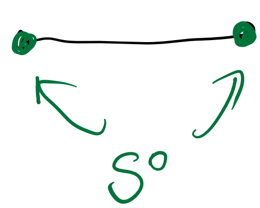
where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

EXAMPLE

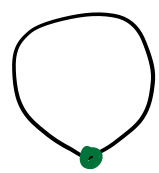
$\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_p(S^n) = 0$ for $i \neq n$,

For $n > 0$ let $(X, A) = (D^n, S^{n-1})$.

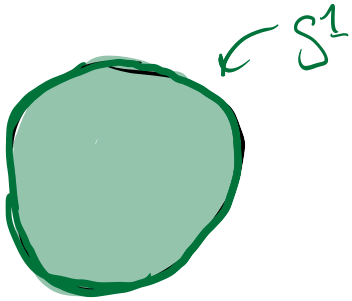
$n = 1$



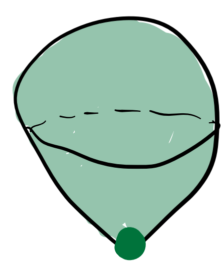
D^1/S^0



$n = 2$



D^2/S^1



For a general n, $D^n / S^{n-1} \approx S^n$.

The LES for homology for (D^n, S^{n-1}) is:

$$\cdots \tilde{H}_p(S^{n-1}) \to \cancel{\tilde{H}_p(D^n)} \to \tilde{H}_p(S^n) \xrightarrow{\cong} \tilde{H}_{p-1}(S^{n-1})$$

$$\to \cancel{\tilde{H}_{p-1}(D^n)} \to \tilde{H}_{p-1}(S^n) \to \cdots \to \tilde{H}_0(D^n) \to \tilde{H}_0(S^n) \to 0$$

D^n is contractible and therefore

$\tilde{H}_p(D^n) \cong \tilde{H}_p(\bullet) = 0 \quad \forall p.$

It follows that $\circledast$ $\tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1})$ for all p. We also know that $\tilde{H}_0(S^n) = 0$.

Let $n = 0$. S^0

We know from theorems in class

that $H_p(X) \cong \bigoplus_{\alpha \in A} H_p(X_\alpha)$, where

X_α for $\alpha \in A$ are the path-connected

components of X. So

$$H_p(S^0) = H_p(\bullet \quad \bullet) \cong H_p(\bullet) \oplus H_p(\bullet)$$

$$\Rightarrow H_p(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & p=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \tilde{H}_p(S^0) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & \text{otherwise} \end{cases}$$

Now we use (*) to get $\tilde{H}_p(S^1) \cong \tilde{H}_{p-1}(S^0)$ for $p > 1$ (from before we know that $\tilde{H}_0(S^1) = 0$).

$$\tilde{H}_p(S^1) = \begin{cases} \mathbb{Z} & p=1 \\ 0 & \text{otherwise} \end{cases}$$

By induction it follows that $\tilde{H}_p(S^n) = \begin{cases} \mathbb{Z} & p=n \\ 0 & \text{otherwise} \end{cases}$