If we have a map

$$f : (X, A) \to (Y, B),$$ then

we also have an induced map

$$f_c : C_n(X, A) \to C_n(Y, B)$$

& $$f_* : H_p(X, A) \to H_p(Y, B) \quad \forall p.$$

(Since f_c takes $C_n(A)$ to $C_n(B)$ the map on quotients is well defined. Also, $f_c \partial = \partial f_c$ holds for relative chains since it holds for absolute chains). We have the following statement about homotopy invariance

PROPOSITION

If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then

$$f_* = g_* : H_n(X, A) \to H_n(Y, B).$$

Proof

Exercise (proof in Hatcher on page 118).

Finally, consider $B \subset A \subset X$.

We have a SES of chain complexes

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0.$$

This sequence induces a LES

$$\cdots H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to$$
$$\to H_{n-1}(A,B) \to \cdots$$

SPLIT EXACT SEQUENCES

Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a SES of abelian groups.

Definition

The sequence is called **SPLIT** if $\exists$ an isomorphism $\tau : B \xrightarrow{\cong} A \oplus C$ s.t. the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & B \arrow[r, "j"] \arrow[d, "\tau", "\cong"'] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A \oplus C \arrow[r, "\pi_c"'] & C \arrow[r] & 0
\end{tikzcd}$$

where $i_A(a) := (a,0)$ and $\pi_c(a,c) := c$.