If we have a map

$$f : (X, A) \to (Y, B), \text{ then}$$

we also have an induced map

$$f_c : S_\bullet(X, A) \to S_\bullet(Y, B)$$

& $\quad f_* : H_p(X, A) \to H_p(Y, B) \quad \forall p.$

(Since f_c takes $S_\bullet(A)$ to $S_\bullet(B)$ the map on quotients is well defined. Also, $f_c \partial = \partial f_c$ holds for relative chains since it holds for absolute chains). We have the following statement about homotopy invariance

PROPOSITION

If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then

$$f_* = g_* : H_n(X, A) \to H_n(Y, B).$$

Proof

Exercise (proof in Hatcher on page 118).

Finally, consider $B \subset A \subset X$.

We have a SES of chain complexes

$$0 \to S_\bullet(A, B) \to S_\bullet(X, B) \to S_\bullet(X, A) \to 0.$$

This sequence induces a LES

$$\cdots H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to$$

$$\to H_{n-1}(A,B) \to \cdots$$

SPLIT EXACT SEQUENCES

Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a SES of abelian groups.

Definition

The sequence is called **SPLIT** if $\exists$ an isomorphism $\tau : B \xrightarrow{\cong} A \oplus C$ s.t. the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & B \arrow[r, "j"] \arrow[d, "\tau", "\cong"'] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A \oplus C \arrow[r, "\pi_C"'] & C \arrow[r] & 0
\end{tikzcd}$$

where $i_A(a) := (a,0)$ and $\pi_C(a,c) := c$.

Proposition

To say that the SES $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is split is equivalent to any of the following three statements

1) $\exists$ a homomorphism $e : B \to B$ with $e \circ e = e$, s.t $\ker e = \operatorname{im} i$.

2) $\exists$ a homomorphism $C \xrightarrow{s} B$ s.t. $j \circ s = id_C$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "j"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

(s is a right inverse to j)

3) $\exists$ a left inverse to i, i.e. a homomorphism $u : B \to A$ with $u \circ i = id_A$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] \arrow[l, bend left, "u"] & C \arrow[r] & 0 \end{tikzcd}$$

Proof

split $\Rightarrow$ 1:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & B \arrow[r, "j"] \arrow[d, "\tau"] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A \oplus C \arrow[r] & C \arrow[r] \arrow[l, bend left, "i_C"] & 0
\end{tikzcd}
$$

Put $e(b) := \tau^{-1} \circ i_C \circ j(b)$

Exercise: check that $e \circ e = e$ and $\ker e = \operatorname{im} i$.

1 $\Rightarrow$ split

Note that $b - e(b) \in \operatorname{Im} i$:

$$e(b - e(b)) = e(b) - e \circ e(b) =$$
$$= e(b) - e(b) = 0$$

$\Longrightarrow b - e(b) \in \ker e = \operatorname{Im}(i)$.

Define $t(b) := (a, j(b))$, where $a \in A$ is the unique element with

$i(a) = b - e(b)$.

Exercise: Check that τ is a homomorphism and that it makes the diagram in the definition commutative

(2) $\Rightarrow$ (1)

Put $e(b) := s \circ j(b)$.

split $\Rightarrow$ (2)

Put $s(c) := \tau^{-1} \circ i_C(c)$.

(3) $\Leftrightarrow$ 1/split

Exercise

$\boxtimes$

Example

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}/2\mathbb{Z} \to 0$$

is a non-split SES, because $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(Alternatively, $\nexists\ \mathbb{Z} \xleftarrow{s} \mathbb{Z}/2\mathbb{Z}$ right inverse to j because s must be 0.)

PROPOSITION

Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a SES of abelian groups. If C is a free abelian, then the sequence splits.

Proof

Let $\{c_\alpha\}_{\alpha \in I}$ be a basis for C.

Define $c : C \to B$ as follows:

$\forall \alpha \in I$ pick $b_\alpha \in j^{-1}(c_\alpha) \subset B$.

not empty since j is surjective

Define $s(c_\alpha) := b_\alpha$.

Now extend linearly to $s : C \to B$. Clearly, $j \circ s = id_C$ ∎

Let $A_\bullet, B_\bullet, \mathcal{C}_\bullet$ be chain complexes, and $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0$ be a SES of chain complexes. The sequence is called SPLIT (in the sense of chain complexes) if $\exists$ a CHAIN MAP $s : \mathcal{C}_\bullet \to B_\bullet$ with $j \circ s = id_{\mathcal{C}_\bullet}$.

$\Leftrightarrow \exists$ a splitting τ with τ being a chain map

$\Leftrightarrow \exists$ u left-inverse of i with u = chain map.

If $\mathcal{A}_\circ$, $\mathcal{C}_\circ$ are chain complexes we can define $\mathcal{A}_\circ \oplus \mathcal{C}_\circ$, where

$$(\mathcal{A}_\circ \oplus \mathcal{C}_\circ)_p = A_p \oplus C_p$$

and the boundary operator is

$$\partial := \partial^A \oplus \partial^B$$

Note that $H_p(\mathcal{A}_\circ \oplus \mathcal{C}_\circ) \cong H_p(\mathcal{A}_\circ) \oplus H_p(\mathcal{C}_\circ)$ because

$$Z_p(\mathcal{A}_\circ \oplus \mathcal{C}_\circ) = Z_p(\mathcal{A}_\circ) \oplus Z_p(\mathcal{C}_\circ)$$
$$B_p(\mathcal{A}_\circ \oplus \mathcal{C}_\circ) = B_p(\mathcal{A}_\circ) \oplus B_p(\mathcal{C}_\circ).$$

If $0 \to \mathcal{A}_\circ \to \mathcal{B}_\circ \to \mathcal{C}_\circ \to 0$ is a split SES of complexes, then

$$H_p(B_\bullet) \cong H_p(A_\bullet) \oplus H_p(C_\bullet) \ \forall p.$$

Remark

Sometimes $0 \to A_p \to B_p \to C_p \to 0$ splits for all p as a sequence of abelian groups, but <u>NOT</u> as a sequence of chain complexes.

Example

X = space, $A \subseteq$ subspace

$$0 \to S_\bullet(A) \xrightarrow{i} S_\bullet(X) \xrightarrow{j} S_\bullet(X, A) \to 0$$

Claim: $\forall p$, $S_p(X, A)$ is free abelian.

A basis for this group: consider $\{\sigma : \Delta^p \to X : \sigma(\Delta^p) \not\subseteq A\} =: \xi$

and $\{j(\sigma)\}_{\sigma \in \xi}$. This family freely generates $S_p(X, A)$.

So, $\forall p$, the sequence

$$0 \to S_p(A) \to S_p(X) \to S_p(X,A) \to 0$$

splits as a sequence of abelian groups, but <u>usually</u> NOT as chain complexes since usually this splitting is not a chain map.

Usually $H_p(X) \not\cong H_p(A) \oplus H_p(X,A)$.

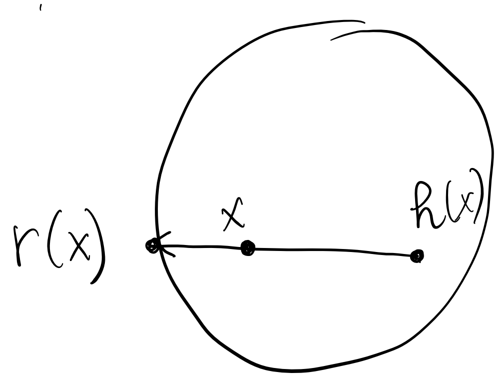
EXAMPLE

Brouwer fixed point theorem

Every continuous map $h: D^n \to D^n$ has a fixed point, that is, a point $x \in D^n$ with $h(x) = x$.

Suppose that $h(x) \neq x \quad \forall x \in D^n$ (proof by contradiction).

then we can define $r: D^n \to S^{n-1}$ by letting $r(x)$ be the point of S^{n-1} where the ray in $\mathbb{R}^n$ starting at $h(x)$ and passing through x leaves D^n.



This map is continuous & $r(x) = x \in S^{n-1}$, or with other words, a retraction.

For $A = S^{n-1}$, $X = D^n$ such an $r: X \to A$ gives a splitting

$$H_p(D^n) \cong H_p(S^{n-1}) \oplus H_p(D^n, S^{n-1}).$$

However, for $p = n-1$

$$H_{n-1}(D^n) = 0,$$

whereas $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, which is not possible.