EXCISION

A fundamental property of relative homology groups is given by the following **EXCISION THEOREM**, describing when the relative groups $H_n(X,A)$ are unaffected by excising/deleting a subset $Z \subset A$.

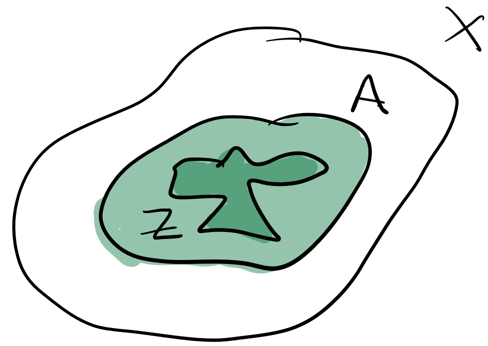
THEOREM (EXCISION)

Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms $H_p(X-Z, A-Z) \to H_p(X,A)$ for all p.

Equivalently, for subspaces $A, B \subset X$ whose interior covers X, the inclusion $(B, A\cap B) \hookrightarrow (X,A)$ induces isomorphisms

$H_p(B, A\cap B) \to H_p(X,A)$ for all p.

The translation between the two versions is obtained by setting



$$B = X - Z \ \& \ Z = X - B.$$

Then $A\cap B = A - Z$ and the condition $\overline{Z} \subset \mathring{A}$ is equivalent to

$$X = \mathring{A} \cup \mathring{B} \text{ since } X - \mathring{B} = \overline{Z}.$$

The proof is quite technical and will be done in several steps.

RELATING HOMOLOGY GROUPS OF A COVERING TO HOMOLOGY GROUPS OF A SPACE

Let X be a space and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a collection of subsets of X s.t.

the interiors of the $\mathcal{U}_\alpha$'s cover X,

$$X = \bigcup_{\alpha \in A} \mathring{\mathcal{U}}_\alpha .$$

We say that a subset $Q \subset X$ is **$\mathcal{U}$-small** if $\exists \alpha \in A$ s.t. $Q \subset \mathcal{U}_\alpha$.

Consider the subgroup of $S_p(X)$ generated by $S_p(\mathcal{U}_\alpha)$ $\forall \alpha$. Denote it by $S_p^{\mathcal{U}}(X)$. The elements are chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$.

The boundary map $\partial : S_p(X) \to S_{p-1}(X)$ takes $S_p^{\mathcal{U}}(X)$ to $S_{p-1}^{\mathcal{U}}(X)$, so the groups $S_p^{\mathcal{U}}(X)$ form a chain complex.

We denote this chain complex $S_\bullet^{\mathcal{U}}(X)$ and it is a subcomplex of $S_\bullet(X)$.

We denote the homology groups of $S^{\mathcal{U}}_{\bullet}(X)$ by $H^{\mathcal{U}}_p(X)$.

THEOREM 1

The inclusion chain map $i^{\mathcal{U}} : S^{\mathcal{U}}_{\bullet}(X) \to S_{\bullet}(X)$ induces an isomorphism in homology

$$i^{\mathcal{U}}_* : H^{\mathcal{U}}_p(X) \xrightarrow{\cong} H_p(X) \quad \forall p.$$

To prove theorem 1, we will apply the so-called barycentric subdivision process.

BARYCENTRIC SUBDIVISION

1 BARYCENTRIC SUBDIVISION OF SIMPLICES

Let $\sigma = [v_0, v_1, \dots, v_n]$ be an n-simplex in $\mathbb{R}^d$. Then

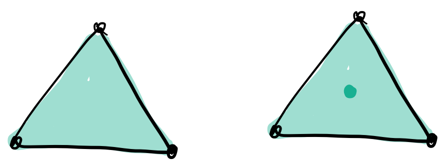
$$\sigma = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; 0 \le t_i \le 1, \ \sum t_i = 1 \right\}$$

The **BARYCENTER** or 'center of gravity' of the simplex σ is the point

$$b = b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

$n=0$ 

$n=1$ 

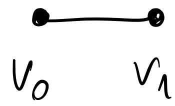
$n=2$ 

The **BARYCENTRIC SUBDIVISION** of $[v_0, v_1, \ldots, v_n]$ is the decomposition of $[v_0, v_1, \ldots, v_n]$ into n-simplices $[b, w_0, \ldots, w_{n-1}]$ where, inductively, $[w_0, \ldots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

The induction starts with $n=0$.

$$n=0 \qquad \bullet \quad \xRightarrow[\text{subdivision}]{\text{barycentric}} \quad \bullet$$

$n=1$

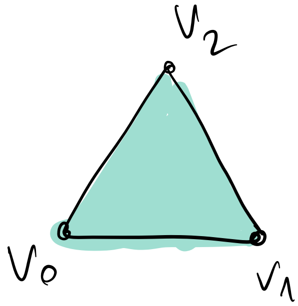


$\xrightarrow{\text{barycentric}}$ subdivision

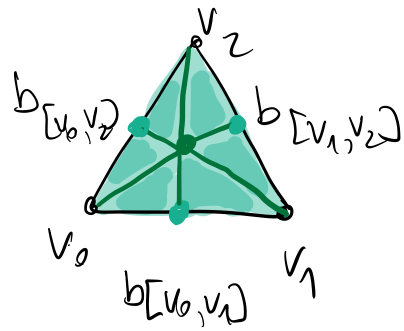
$[b, v_0]$

$[b, v_1]$

$n=2$



$\xrightarrow{\text{barycentric}}$ subdivision



$n \geq 2$

$\sigma = [v_0, \dots, v_n]$ $\xrightarrow{\text{barycentric}}$ subdivision n-simplices

$\tau = [b, w_0, \dots, w_{n-1}]$

where $[w_0, \dots, w_{n-1}]$ is a $(n-1)$-sx from the barycentric subdivision of a face $[v_0, \dots, \hat{v_i}, \dots, v_n]$ of σ

CLAIM

$$\mathrm{diam}[b, w_0, \dots, w_{n-1}] \leq \frac{n}{n+1} \mathrm{diam}[v_0, \dots, v_n]$$

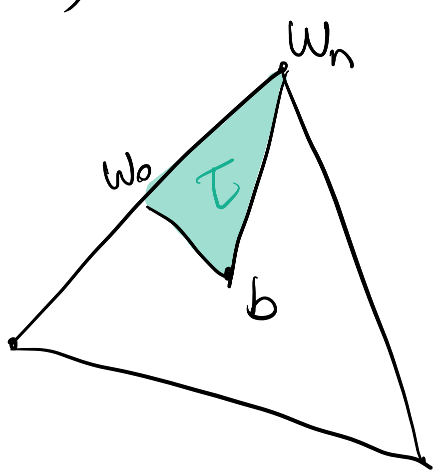
max distance between any two of its vertices since

$$|v - \sum_i t_i v_i| = |\sum_i t_i (v - v_i)| \leq \sum_i t_i |v - v_i|$$

$$\leq \sum_{i} t_i \max_j |v - v_j| = \max_j |v - v_j|$$

To obtain the bound, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex τ of the barycentric subdivision of $[v_0, \dots, v_n]$ is at most

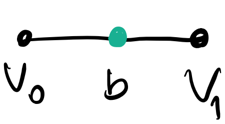
$$\frac{n}{(n+1)} \operatorname{diam}[v_0, \dots, v_n].$$



① w_j & $w_k \neq b$, the barycenter of $[v_0, \dots, v_n]$

In this case the statement follows by induction on n as these two points lie in a proper face of $[v_0, \dots, v_n]$:

$n = 1$



$$\begin{matrix} \operatorname{diam}[v_0, b] \\ \shortparallel \\ \operatorname{diam}[v_1, b] \end{matrix} \leq \frac{1}{2} \operatorname{diam}[v_0, v_1]$$

$$|w_i - w_j| \overset{IH}{\leq} \frac{n-1}{n} \operatorname{diam}[v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\leq \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$$

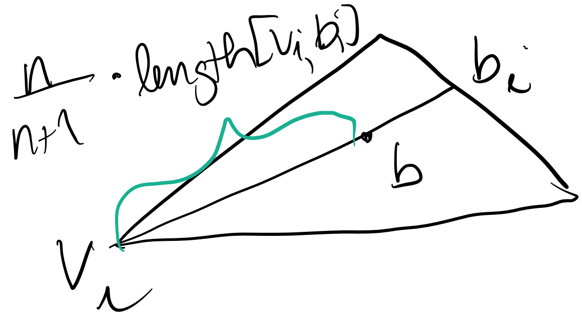
$$\frac{n-1}{n} \leq \frac{n}{n+1}$$

(2) Suppose WLOG that $w_j = b$.
then

$|b - w_k| \leq |b - v_i|$ for some i.

Let b_i be the barycenter of $[v_0, \dots, \hat{v}_i, \dots, v_n]$

$$b_i = \frac{1}{n} \sum_{j \neq i} v_j$$



$$b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$$

$$|b - v_i| = \frac{n}{n+1} |b_i - v_i|$$

IMPORTANT: $\left(\frac{n}{n+1}\right)^r \xrightarrow[r\to\infty]{} 0$

$$\leq \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$$

(2) BARYCENTRIC SUBDIVISION OF LINEAR CHAINS

Let $Y \subset \mathbb{R}^d$ be a convex set.

We define

$$LS_p(Y) = \langle \sigma : \Delta^p \to Y \mid \sigma \text{ is a linear map} \rangle$$

linear simplices in Y

$$\sigma\left(\sum_{i=0}^{p} t_i e_i\right) = \sum_{i=0}^{p} t_i \sigma(e_i)$$

standard basis

$LS_p(Y) \subset S_p(Y)$ & the boundary map

maps $LS_p(Y)$ to $LS_{p-1}(Y)$.

Let $LS_{-1}(Y) = \mathbb{Z}\langle[\phi]\rangle \leftarrow$ empty simplex

and $\partial[w_0] = [\phi]$ $\forall$ 0-sx w_0.

We have the following chain complex,

$$\cdots \to LS_p(Y) \to LS_{p-1}(Y) \to \cdots \to LS_1(Y) \to LS_0(Y) \to \mathbb{Z} \to \cdots$$

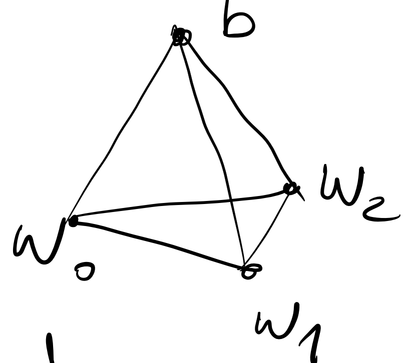
a subcomplex of $S_\bullet(Y)$ that we denote by $LS_\bullet(Y)$.

Each $b \in Y$ determines a homomorphism $b : LS_p(Y) \to LS_{p+1}(Y)$ defined by :

$$b([w_0, \ldots, w_p]) = [b, w_0, \ldots, w_p]$$

& extended to all of $LS_p(Y)$ linearly.

CONE OPERATOR



b sends a linear chain to the cone that has this chain as a base & whose tip is b

Let's compute

$$\partial (b[w_0, \dots, w_p]) = \partial ([b, w_0, \dots, w_p])$$

$$= (-1)^0 [w_0, \dots, w_p] + (-1)^1 [b, w_1, \dots, w_p]$$

$$+ (-1)^2 [b, w_0, w_2, \dots, w_p] + \dots + (-1)^p [b, w_0, \dots, w_{p-1}]$$

$$= [w_0, \dots, w_p] - b ([w_1, \dots, w_p] + (-1)^1 [w_0, w_2, \dots, w_p]$$

$$+ \dots + (-1)^p [w_0, \dots, w_{p-1}]) =$$

$$= [w_0, \dots, w_p] - b \, \partial [w_0, \dots, w_p] =$$

$$= (id - b \circ \partial) [w_0, \dots, w_p]$$

$$\Rightarrow \partial b = id - b \circ \partial$$

b is a **CHAIN HOMOTOPY** between 0 and the identity on the augmented chain complex $LS_\bullet(Y)$.