Now we define a **SUBDIVISION HOMOMORPHISM** $sd_p : LS_p(Y) \to LS_p(Y)$

by induction on p.

$p = -1$

$$sd_{-1}([\phi]) = [\phi]$$

$$sd_{-1} = id$$

$p \geq 0$ for generators $\sigma \in LS_p(Y)$

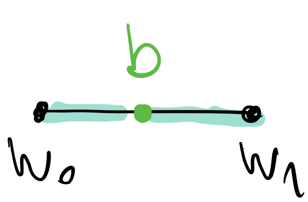
$$sd_p(\sigma) = b_\sigma(sd_{p-1}(\partial\sigma)),$$

where b_σ is the barycenter of σ

$p = 0$ $\quad sd_0([w_0]) = w_0([\phi]) = [w_0]$

$$\Rightarrow sd_0 = id$$

$p = 1$



$$sd_1([w_0, w_1]) =$$

$$= b((-1)^0[w_1] + (-1)^1[w_0])$$

$$= b([w_1] - [w_0]) =$$

$= [b, w_1] - [b, w_0]$

sum of the 1-simplices in the barycentric subdivision with certain signs

(compare the def. of sd with that of the subdivision of a simplex)

sd is a chain map.

We prove this by induction:

$$\underbrace{sd_{-1}}_{= \mathrm{id}} \circ \partial = \partial \circ \underbrace{sd_0}_{= \mathrm{id}} \quad \checkmark$$

$$\begin{array}{ccccccccccc}
\cdots & LS_{p+1}(Y) & \to & LS_p(Y) & \to \cdots \to & LS_0(Y) & \to & LS_{-1}(Y) & \to & 0 \\
 & \downarrow sd_{p+1} & & \downarrow sd_p & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
\cdots & LS_{p+1}(Y) & \to & LS_p(Y) & \to \cdots \to & LS_0(Y) & \to & LS_{-1}(Y) & \to & 0
\end{array}$$

Squares up to p commute.

$$\partial b_\sigma = id - b_\sigma \partial$$

induction step:

$$\partial(sd_{p+1}(\sigma)) = \partial(b_\sigma(sd_p(\partial\sigma))) =$$

$$\partial sd_p = sd_{p-1}\partial$$

$$= sd_p(\partial\sigma) - b_\sigma(\partial sd_p(\partial\sigma)) =$$

$$\partial\partial = 0$$

$$= sd_p(\partial\sigma) - b_\sigma(\cancel{sd_{p-1}\partial(\partial\sigma)}) =$$

$$= sd_p(\partial\sigma)$$

Next we build a chain homotopy $D: LS_p(Y) \to LS_{p+1}(Y)$ between sd and id.

chain map

$$D_p: LS_p(Y) \to LS_{p+1}(Y)$$

s.t. $\partial D + D\partial = id - sd$

We define D inductively.

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] \arrow[d, "sd"', "id"] & LS_{-1}(Y) \arrow[r] \arrow[d, "sd"', "id"] \arrow[dl, "D_{-1}"'] & 0 \arrow[dl, "D_{-2}"] \\
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] & LS_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

$D_{-2} = 0$

$D_{-1} = 0$

$$\partial D_{-1} + D_{2}\partial = 0 + 0 = 0$$

$$id - sd_{-1} = id - id = 0$$

$$\Rightarrow \partial D_{-1} + D_{-2}\partial = id - sd_{-1}$$

$p \geq 0$ We define D_p inductively.

$\sigma \in LS_p(Y)$ a simplex

$$D_p(\sigma) = b_\sigma\left(\sigma - D_{p-1}(\partial\sigma)\right)$$

barycenter of σ

We check using induction that $\partial D + D\partial = id - sd$. Assume that all maps up to D_p satisfy this.

$$\partial D_{p+1}\sigma = \partial\left(b_\sigma\left(\sigma - D_p(\partial\sigma)\right)\right) \overset{\curvearrowleft}{=} \quad \partial b_\sigma = id - b_\sigma\partial$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma\left(\partial\left(\sigma - D_p(\partial\sigma)\right)\right)$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma\left(\partial\sigma - \partial D_p(\partial\sigma)\right)$$

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r] \arrow[d] & LS_{p+1}(Y) \arrow[r, "\partial"] \arrow[d, "sd"', "id"] \arrow[dl, "D_{p+1}"'] & LS_p(Y) \arrow[r] \arrow[d, "sd"', "id"] \arrow[dl, "D_p"'] & LS_{p-1}(Y) \arrow[r] \arrow[d] \arrow[dl, "D_{p-1}"'] & \cdots \\
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r, "\partial"'] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & LS_{p-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

I.H: $\partial D_p + D_{p-1}\partial = id - sd$

$\implies id - \partial D_p = sd_p + D_{p-1}\partial$

$$\overset{I.H}{=} \sigma - D_p(\partial\sigma) - b_\sigma\left(sd_p(\partial\sigma) + D_{p-1}\partial(\partial\sigma)\right)$$

$$= \sigma - D_p\partial(\sigma) - b_\sigma\, sd_p(\partial\sigma)$$

$$= \sigma - D_p\partial(\sigma) - sd_{p+1}(\sigma)$$

$$\implies \partial D_{p+1}(\sigma) + D_p\partial(\sigma) = \sigma - sd_{p+1}(\sigma)$$

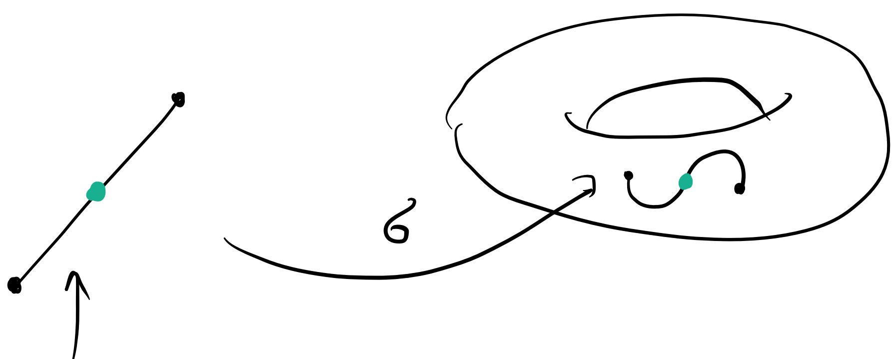
③ BARYCENTRIC SUBDIVISION OF GENERAL CHAINS

Let X be a topological space.

Define homomorphisms

$$sd = sd_p : S_p(X) \to S_p(X)$$

on generators $\sigma(\sigma : \Delta^p \to X)$



We subdivide this space, which is a convex subset in $\mathbb{R}^{p+1}$

$$sd(\sigma) = \sigma_c\big(sd(\underbrace{id : \Delta^p \to \Delta^p}))$$

$$(id : \Delta^p \to \Delta^p) \in LS_p(\Delta^p) \subset S_p(\Delta^n)$$

$\sigma : \Delta^p \to X$ induces $\sigma_c : S_p(\Delta^p) \to S_p(X)$.

$$\begin{tikzcd} \mathrm{id} \in S_p(\Delta^p) \arrow[r, "\sigma_c"] \arrow[d, "sd"] & S_p(X) \ni \sigma \arrow[d, "sd"] \\ S_p(\Delta^p) \arrow[r, "\sigma_c"] & S_p(X) \end{tikzcd}$$

$S_p(\Delta^p) \ni sd(id)$

sd is a chain map

$$\partial(sd(\sigma)) = \partial\,\sigma_c\,(sd(id : \Delta^p \to \Delta^p)) =$$

$$= \sigma_c\,\partial\,(sd(id : \Delta^p \to \Delta^p)) =$$

σ_c is a chain map

$$= \sigma_c\, sd(\partial\, id_{\Delta^p}) =$$

$$= \sigma_c\, sd\left(\sum_{i=0}^{p} (-1)^i\, id_{\Delta_i^p}\right)$$

restriction of id to the ith face of Δ^p

$$= \sum_{i=0}^{p} (-1)^i \sigma_c \, sd(id_{\Delta_i^p})$$

sum of signed simplices in the barycentric subdivision of Δ_i^p

$$= \sum_{i=0}^{p} (-1)^i \, sd\left(\sigma_c \big|_{\Delta_i^p}\right)$$

$$= sd\left(\sum_{i=0}^{p} (-1)^i \sigma_c \big|_{\Delta_i^p} \right) =$$

$$= sd(\partial \sigma)$$

In a similar fashion we define

$D : S_p(X) \longrightarrow S_{p+1}(X)$

$D(\sigma) = \sigma_c\left(D(id_{\Delta^p})\right)$

here we take the D defined for singular chains

D is a chain homotopy between sd & id.

σ_c is a chain map

$$\partial D(\sigma) = \partial\left(\sigma_c\left(D(id_{\Delta^p})\right)\right) =$$

$$= \sigma_c\left(\partial D(id_{\Delta^p})\right) =$$

D is a chain homotopy for linear chains

$$= \sigma_c \left(id_{\Delta^p} - sd(id_{\Delta^p}) - D\partial(id_{\Delta^p}) \right)$$

$$\sigma_c D\partial(id_{\Delta^p}) = \sigma_c D\left(\sum_{i=0}^{p}(-1)^i id_{\Delta_i^p}\right) = \sum_{i=0}^{p}(-1)^i \sigma_c D\, id_{\Delta_i^p} = \sum_{i=0}^{p}(-1)^i D(\sigma_i) = D\sum_{i=0}^{p}(-1)^i \sigma_i = D\,\partial\sigma$$

$$= \sigma - sd(\sigma) - D\partial(\sigma) =$$

$$= (id - sd - D\partial)(\sigma)$$

Before we prove Theorem 1, let's recall

Lebesgue's number Lemma:

If the metric space (X,d) is compact & an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover.

PROOF OF THEOREM 1

Let $\mathcal{U}$ be a covering as in the statement of Theorem 1. Let $\sigma \in Sp(X)$ be a singular simplex.

Then $\{ \sigma^{-1}(\mathring{U}) \mid U \in \mathcal{U} \}$ is an open covering of Δ^p. Δ^p is compact, so we can select the Lebesgue number δ of this covering.

Pick $m \in \mathbb{N}$ large enough that

$$\left(\frac{p}{p+1}\right)^{m_\delta} \underbrace{\sqrt{2}}_{\text{diameter of an } p\text{-simplex}} \leq \delta.$$

m will determine how much we have to subdivide simplices so that each lies in some $U \in \mathcal{U}$

If we use sd m-times on

σ we get a chain consisting of singular simplices, of which each lies in some $U \in \mathcal{U}$.

$\sigma_c(sd^m(id_{\Delta^p})) = sd^m(\sigma) \in S_p^{\mathcal{U}}(X)$.

For each p-simplex σ we select the m_σ in a way that it is the smallest non-negative integer for which $sd^{m_\sigma}(\sigma) \in S_p^{\mathcal{U}}(X)$

$(m_\sigma = 0 \Leftarrow\Rightarrow \sigma \in S_p^{\mathcal{U}}(X))$.

We define

$$\overline{D} : S_p(X) \to S_{p+1}(X)$$

$$\overline{D}(\sigma) = \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma))$$

this is the D that we defined for singular chains

for σ a p-simplex

If $m_\sigma = 0$, $\overline{D}(\sigma) = 0$.

We calculate

$$(\partial\overline{D} + \overline{D}\partial)(\sigma) = \partial \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma))$$

$$+ \sum_{i=0}^{p} (-1)^i \,\overline{D}\, \sigma^i =$$

i-th face of σ

$$= \sum_{j=0}^{m_\sigma-1} \partial D(sd^j(\sigma)) + \sum_{i=0}^{p}(-1)^i \sum_{j=0}^{m_{\sigma_i}-1} D(sd^j(\sigma^i))$$

$$= \sum_{j=0}^{m_\sigma-1} \left(sd^j(\sigma) - sd^{j+1}(\sigma) - D\partial(sd^j(\sigma))\right)$$

$$+ \sum_{i=0}^{p}(-1)^i \sum_{j=0}^{m_{\sigma_i}-1} D(sd^j(\sigma^i)) =$$

$$= \sigma - sd^{m_\sigma}(\sigma) - \sum_{j=0}^{m_\sigma-1} D(sd^j(\partial\sigma)) +$$

$$+\sum_{i=0}^{p}(-1)^i\sum_{j=0}^{m_{\sigma_i}-1} D(sd^j(\sigma^i))=$$

$$=\sigma - sd^{m_\sigma}(\sigma) - \sum_{j=0}^{m_\sigma-1}\sum_{i=0}^{p}(-1)^i D(sd^j(\sigma_i))$$

$$+\sum_{i=0}^{p}(-1)^i\sum_{j=0}^{m_{\sigma_i}-1} D(sd^j(\sigma^i)) =$$

$$=\sigma - sd^{m_\sigma}(\sigma) + \sum_{i=0}^{p}(-1)^i\sum_{j=m_{\sigma_i}}^{m_\sigma-1} D(sd^j(\sigma^i))$$

$$(\ m_{\sigma_i}\leq m_\sigma\ )$$

We set

$$\rho(\sigma):=\sigma-\partial\overline{D}(\sigma)-\overline{D}\partial(\sigma)$$

Note that $\rho(\sigma)\in S_p^{\mathcal{U}}(X)$.

This ρ is a map: $S_p(X)\to S_p^{\mathcal{U}}(X)$.

ρ is a chain map:

$$\partial\rho(\sigma) = \partial\sigma - \partial\partial\bar{D}(\sigma) - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma) - \bar{D}\partial\partial(\sigma)$$

$$= \rho(\partial\sigma)$$

$\Rightarrow \partial\bar{D} - \bar{D}\partial = id - i_c^u \circ \rho$,
where $i_c^u : S_\bullet^u(X) \to S_\bullet(X)$ is the inclusion.

$\bar{D}$ is a chain homotopy from $i_c^u \circ \rho$ to id.

Also, $\rho \circ i_c^u = \rho(i_c^u(\sigma)) =$

$$= \sigma - \underbrace{\partial\bar{D}(i_c^u(\sigma))}_{=0} - \underbrace{\bar{D}\partial(i_c^u(\sigma))}_{=0}$$

$= id$

so ρ is the chain homotopy inverse of $i_c^{\mathcal{U}}$.

It follows from homotopy invariance statements that $i_*^{\mathcal{U}}$ is an isomorphism $H_p^{\mathcal{U}}(X) \xrightarrow{i_*^{\mathcal{U}}} H_p(X)$.

PROOF OF EXCISION THEOREM

Let $\mathcal{U} = \{A, B\}$ such that $\mathring{A} \cup \mathring{B} = X$,

$$i_c^{\mathcal{U}} : S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$$

is a chain equivalence. From proof of theorem 1 we get maps ρ & $\overline{D}$ that map simplices in A to simplices in A.