$= id$

so ρ is the chain homotopy inverse of $i_c^{\mathcal{U}}$.

It follows from homotopy invariance statements that $i_*^{\mathcal{U}}$ is an isomorphism $H_p^{\mathcal{U}}(X) \xrightarrow{i_*^{\mathcal{U}}} H_p(X)$.

PROOF OF EXCISION THEOREM

Let $\mathcal{U} = \{A, B\}$ such that $\mathring{A} \cup \mathring{B} = X$.

$$i_c^{\mathcal{U}} : S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$$

is a chain equivalence. From proof of theorem 1 we get maps ρ & $\overline{D}$ that map simplices in A to simplices in A.

ρ and $\overline{D}$ induce maps on quotients

$$\rho : S_p(X) / S_p(A) \to S_p^{\mathcal{U}}(X) / S_p(A)$$

$$\overline{D} : S_p(X) / S_p(A) \to S_{p+1}(X) / S_{p+1}(A)$$

It still holds that

$$\partial \overline{D} + \overline{D} \partial = id - \iota_c^{\mathcal{U}} \circ \rho$$

and that

$$\iota_c^{\mathcal{U}} : S_\bullet^{\mathcal{U}}(X) / S_\bullet(A) \to S_\bullet(X) / S_\bullet(A)$$

is a chain equivalence and consequently it induces an isomorphism on homology.

The map

$$\frac{S_p(B)}{S_p(A\cap B)} \longrightarrow \frac{S_p^{\mathcal{U}}(X)}{S_p(A)}$$

induced by inclusion is an isomorphism since both quotient groups are free with the basis singular p-simplices in B that do not lie in A. $\Rightarrow$

$$H_p(X,A) \cong H_p\left(\frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet(A)}\right)$$

$$\cong H_p(B, A\cap B).$$

$\blacksquare$

Here is an example of the machinery we developed, a classical result from 1910 due to Brouwer, known as

INVARIANCE OF DIMENSION

If non-empty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Let $x \in U$. By excision

$$H_p(U, U-\{x\}) \cong H_p(\mathbb{R}^m, \mathbb{R}^m-\{x\}).$$

From LES of $(\mathbb{R}^m, \mathbb{R}^m-\{x\})$

$$\cdots \tilde{H}_p(\mathbb{R}^m-\{x\}) \to \tilde{H}_p(\mathbb{R}^m) \to H_p(\mathbb{R}^m, \mathbb{R}^m-\{x\}) \to$$

$$\to \tilde{H}_{p-1}(\mathbb{R}^m-\{x\}) \to \tilde{H}_{p-1}(\mathbb{R}^m) \to$$

we get $\quad H_p(\mathbb{R}^m, \mathbb{R}^m-\{x\}) \cong \tilde{H}_{p-1}(\mathbb{R}^m-\{x\})$

Since $\mathbb{R}^m-\{x\}$ strongly deformation retracts to S^{m-1},

$$H_p(U, U-\{x\}) \cong H_{p-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & p=m \\ 0 & \text{otherwise} \end{cases}$$

Homeomorphism $h: U \to V$ yields a homeomorphism of pairs

$$(U, U-\{x\}) \text{ and } (V, V-\{h(x)\})$$

and so

$$H_p(U, U-\{x\}) \cong H_p(V, V-\{h(x)\}).$$

Since also

$$H_p(V, V-\{h(x)\}) \cong H_{p-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & p=n \\ 0 & \text{otherwise} \end{cases}$$

it follows that $m=n$.