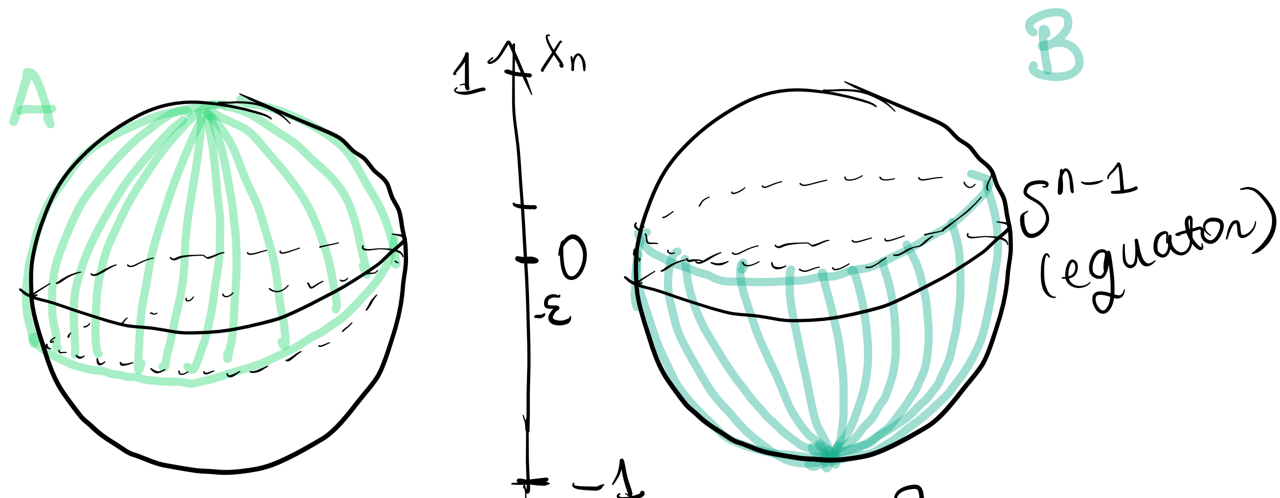
EXAMPLE (special case of the previous problem)

$X = S^n, \ n \geq 1. \qquad \cong \tilde{H}_{p-1}(X) \oplus \tilde{H}_{p-1}(X_2)$

$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ ; \ x_0^2 + \dots + x_n^2 = 1\}$



$A = \{(x_0, \dots, x_n) \in S^n : -\varepsilon < x_n \leq 1\}$

$B = \{(x_0, \dots, x_n) \in S^n : -1 \leq x_n < \varepsilon\}$

$A, B \approx (\mathring{B}^n)$ & so A, B are contractible

$\Rightarrow \tilde{H}_p(A) = \tilde{H}_p(B) = 0 \quad \forall p.$

Also, $A \cap B \simeq S^{n-1}$ ($A \cap B \neq \phi$ since $n \geq 1$).

We now use the Mayer-Vietoris LES:

$$\cdots \to \tilde{H}_p(S^{n-1}) \to 0 \oplus 0 \to \tilde{H}_p(S^n) \to \tilde{H}_{p-1}(S^{n-1}) \to 0 \oplus 0 \to$$

$$\Rightarrow \tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1}).$$

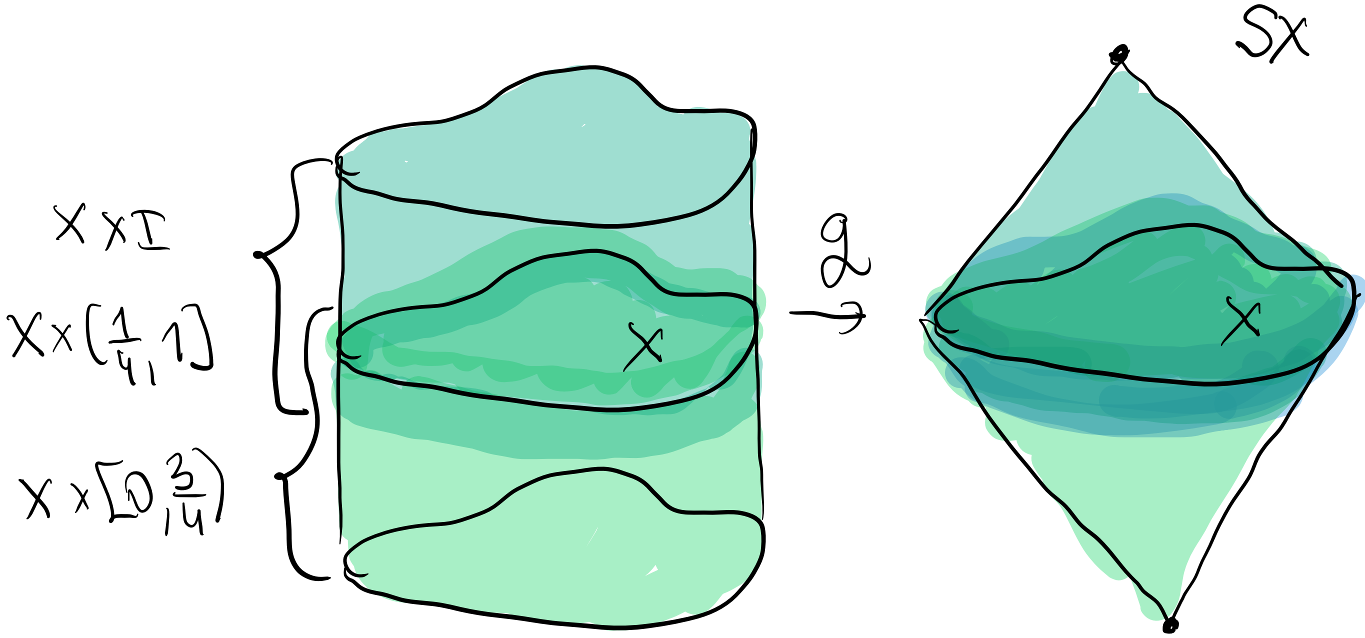
$$\Rightarrow \tilde{H}_p(S^n) \cong \cdots \cong \tilde{H}_{p-n}(S^0) = \begin{cases} \mathbb{Z} & p=n \\ 0 & p\neq n \end{cases}$$

EXAMPLES

Show that $\tilde{H}_p(X) \cong \tilde{H}_{p+1}(SX)$ for all p, where SX is the suspension of X.

More generally, thinking of SX as the union of two cones CX with their base identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified. $(n=3)$

Recall that the suspension of X, SX, is $X \times I$ with $X \times \{0\}$ and $X \times \{1\}$ collapsed into a point.



Let $A = g\left(X \times (\tfrac{1}{4}, 1]\right)$ The interiors of A & B cover SX.

$B = g\left(X \times [0, \tfrac{3}{4})\right)$.

A & B are both contractible and therefore have vanishing reduced homology

$A \cap B$ is homotopy equivalent to X.

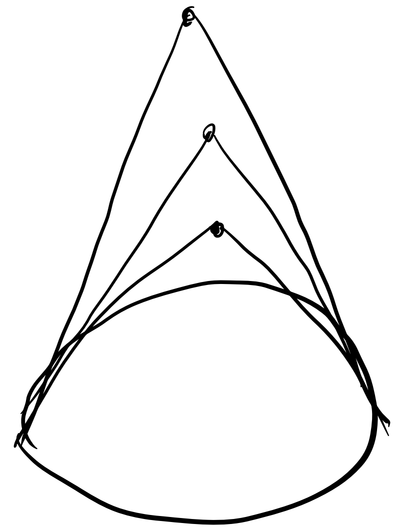
Then the reduced Mayer-Victoris sequence yields

$$\to \underbrace{\tilde{H}_p(A) \oplus \tilde{H}_p(B)}_{=0} \to \tilde{H}_p(SX) \to \tilde{H}_{p-1}(X) \to \underbrace{\tilde{H}_{p-1}(A) \oplus \tilde{H}_{p-1}(B)}_{=0} \to$$

and hence $\tilde{H}_p(SX) \cong \tilde{H}_{p-1}(X)$.

Now consider $n=3$

We denote this space by S_3X.



We use the Mayer-Vietoris sequence for

$$S_3X = SX \cup_X CX :$$

both slistlly thickened

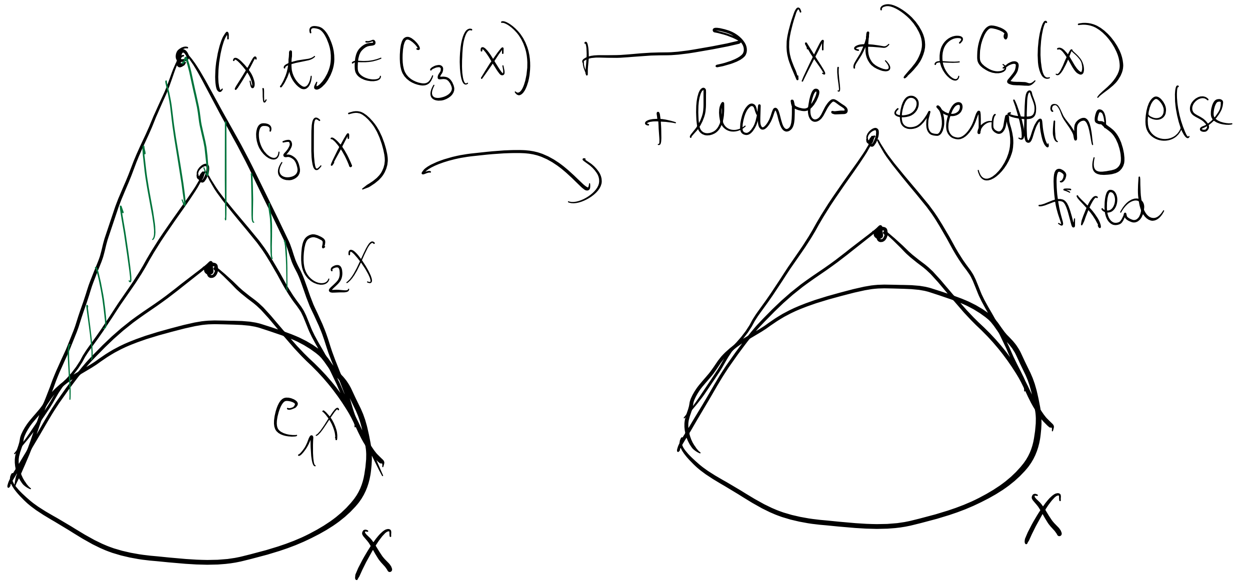
$$\dots \to \tilde{H}_p(X) \to \tilde{H}_p(SX) \oplus \tilde{H}_p(CX) \to \tilde{H}_p(S_3X) \to \dots$$

We have $\tilde{H}_p(CX) = 0$ for all p.

Furthermore, the morphism induced by inclusion $\tilde{H}_p(X) \to \tilde{H}_p(SX)$ is trivial since any cycle in X is a boundary inside SX (boundary of the cone, for example). Hence, this sequence simplifies to

$$0 \to \tilde{H}_p(SX) \to \tilde{H}_p(S_3 X) \to \tilde{H}_{p-1}(X) \to 0 \quad \forall p$$

Now observe that there exists a retraction $r : S_3 X \to SX$.



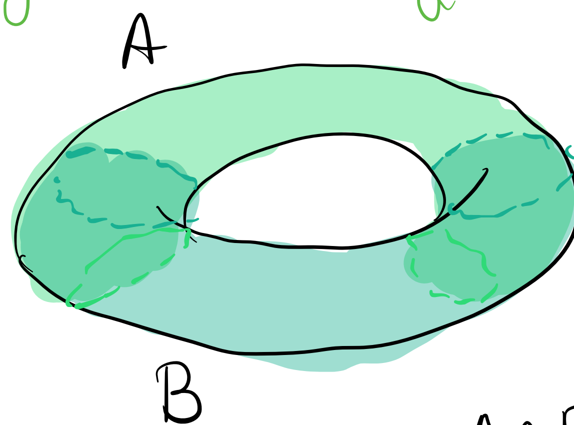
this retraction induces a map $r_* : \tilde{H}_p(S_3 X) \to \tilde{H}_p(SX)$ with

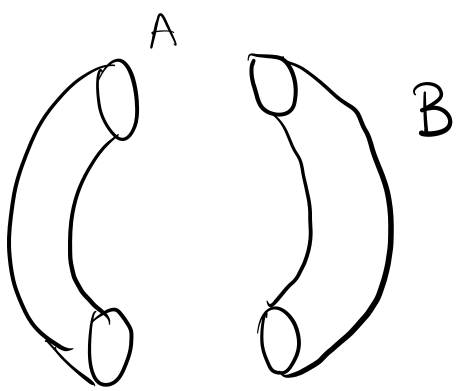
$r_* \circ i_* = id_{\tilde{H}_p(S_p)}$, where $i : SX \hookrightarrow S_3 X$ is an inclusion. This means that this SES is split. In particular,

$$\tilde{H}_p(S_3 X) \cong \tilde{H}_{p-1}(X) \oplus \tilde{H}_p(SX)$$

EXAMPLE

Compute homology groups of the torus using the Mayer-Vietoris sequence.





$A \cap B$ homotopy equivalent to $S^1 \sqcup S^1$

A, B homotopy equivalent to S^1

The Mayer-Vietoris sequence is torus

$$\cdots H_p(A \cap B) \to H_p(A) \oplus H_p(B) \to H_p(T) \to H_{p-1}(A \cap B) \to \cdots$$

For $p > 2$ we get

$$\cdots \to 0 \oplus 0 \to H_p(T) \to 0 \to \cdot$$

$\tilde{H}_p(S^1)=0 \quad p\geq 2$

and therefore $H_p(T)=0$ for $p>2$.

For $p=0$ consider

$$H_2(A)\oplus H_2(B)\to H_2(T^2)\to H_1(A\cap B)\to H_1(A)\oplus H_1(B)$$

$$\to H_1(T^2)\to \tilde{H}_0(A\cap B)\to \tilde{H}_0(\ )\oplus \tilde{H}_0(\ )$$

$$\to \tilde{H}_0(T)\to 0$$

From here we get $\tilde{H}_0(T)=0$. Now consider

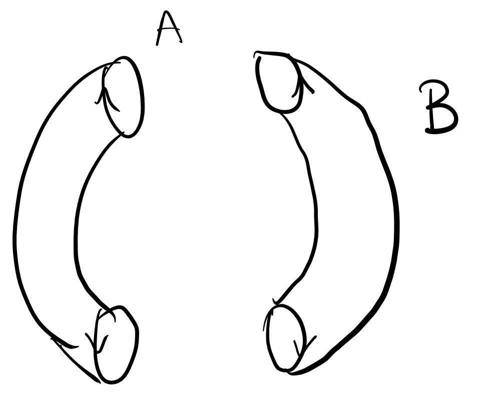
$$0\to H_2(T^2)\xrightarrow{\partial} H_1(A\cap B)\xrightarrow{(i^A_*,-i^B_*)} H_1(A)\oplus H_1(B)$$

$$\to H_1(T^2)\xrightarrow{\partial_*} H_0(A\cap B)\to 0.$$

Let us first compute $H_2(T^2)$.

∂ is injective, so $H_2(T^2)$ is isomorphic to $\operatorname{Im}\partial_*$. By exactness, $\operatorname{Im}\partial_*=\ker(i^A_*,-i^B_*)$. So the next step is to determine $\ker(i^A_*,-i^B_*)$.

For this we choose the cycles generating the homologies of A, B & $A \cap B$.



$A \cap B$ homotopy equivalent to $S^1 \sqcup S^1$

Now $H_1(A \cap B) \cong \langle \alpha \rangle \oplus \langle \beta \rangle$.

In $H_1(A)$ & $H_1(B)$ $\alpha = \beta$, so

$$(i_*^A, -i_*^B)(\alpha, 0) = (i_*^A, -i_*^B)(0, \beta) = (\alpha, -\beta)$$

Hence, $(i_*^A, -i_*^B)$ can be represented by

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

Hence, $H_2(T) = \operatorname{Im} \partial_* = \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} =$

$$= \langle \alpha - \beta \rangle = \mathbb{Z}$$

For $H_1(T^2)$ we focus on the following piece of the MVS:

$$\cdots H_1(A \cap B) \xrightarrow{(i^A_*, -i^B_*)} H_1(A) \oplus H_1(B) \xrightarrow{j^*_A + j^*_B} H_1(T^2) \xrightarrow{\partial_*} H_0(A \cap B) \to \cdots$$

We know all the groups except $H_1(T^2)$. Making an argument similar to the previous one, we can show that

$$(i^A_* - i^B_*) : H_0(A \cap B) \to H_0(A) \oplus H_0(B)$$

can be represented by $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

Now we produce a SES from the sequence above:

$$0 \to \ker \partial_* \to H_1(T^2) \to \operatorname{Im} \partial_* \to 0$$

$$\operatorname{Im} \partial_* = \ker((i^A_*, -i^B_*)) = \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \mathbb{Z}$$

We also know that

second iso theorem

$$\ker \partial_* = \operatorname{Im}(j_A^* + j_B^*) \cong \mathbb{Z} \oplus \mathbb{Z} \Big/ \ker(j_A^* + j_B^*)$$

$$= \mathbb{Z} \oplus \mathbb{Z} \Big/ \operatorname{Im}(i_*^A, -i_*^B).$$

$$\cong \mathbb{Z}$$

Hence, we have the following split SES ($\mathbb{Z}$ is free, so we have a right inverse)

$$0 \to \mathbb{Z} \to H_1(T) \to \mathbb{Z} \to 0$$

thus $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

So, for the torus, the homology groups are

$$H_i(T^2) = \begin{cases} 0 & i \geq 2 \\ \mathbb{Z} & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}$$