There is also a version of Mayer-Vietoris for closed subsets

MAYER-VIETORIS SEQUENCE FOR CLOSED SUBSETS (MV#2)

Let X be a space & A, B closed subsets of X s.t. $X = A \cup B$. Also assume that A is a strong deformation retract of its neighborhood in X U, B is a strong deformation retract of its neighborhood V and $A \cap B$ is a strong deformation retract of $U \cap V$. Then

$$\cdots \to H_p(A \cap B) \xrightarrow{i_*^A \oplus (-i_*^B)} H_p(A) \oplus H_p(B) \xrightarrow{j_*^A + j_*^B} H_p(X) \to$$
$$\to H_{p-1}(A \cap B) \to \cdots$$

There is also such a LES for reduced hom.

THE MV SEQUENCE FOR THE RELATIVE CASE (Hatcher, 152) (MV#3)

Let X be a space & $A, B \subset X$. Put $Y = A \cup B$, viewed as a subspace of X. Assume $A, B \subset Y$ are open. Then $\exists$ a LES:

$$\cdots H_p(X, A\cap B) \xrightarrow{i_*^{(X,A)} \oplus -i_*^{(X,B)}} H_p(X,A) \oplus H_p(X,B) \xrightarrow{j_*^{(X,A)} + j_*^{(X,B)}} H_p(X, A\cup B) \to H_{p-1}(X, A\cap B) \to \cdots$$

We will prove MV#2. In order to do it, we need the following results we already mentioned in the homological algebra section.

FIVE-LEMMA (#2)

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "l'"'] & E'
\end{tikzcd}$$

and the rows are exact sequences, then γ is an isomorphism.

Proof (for completeness)

① γ is surjective

Take $c' \in C'$. Then $k'(c') = \delta(d)$ for some $d \in D$. By exactness

$$0 = l'(k'(c')) = l'(\delta(d)) = \varepsilon l(d)$$

Since ε is an isomorphism,

$l(d) = 0$. So $d \in \ker l = \operatorname{Im} k$.

This means that $c \in C$ exists

such that $k(c) = d$.

$m(c) \in C'$. Also

$$k'(m(c) - c') = k'm(c) - k'(c) =$$
$$= \delta k(c) - \delta(d)$$
$$= 0$$

$\Rightarrow m(c) - c' \in \ker k'$. Since $\ker k' = \operatorname{Im} j'$,

b' exists such that

$c - m(c)$

$$j'(b') = m(c) - c'.$$

$\exists b \in B$ with $\beta(b) = b'$.

$$m(c) - c' = j'\beta(b) = mj(b)$$
$$m(c - j(b)) = c'$$

② m is injective

$m(c) = 0$ implies that

$$0 = k' m(c) = \delta k(c).$$

Since δ is an isomorphism, $k(c) = 0$.
So $c \in \operatorname{Ker} k = \operatorname{Im} j$ and $b \in B$ exists such that

$$j(b) = c.$$

$$0 = m j(b) = j' \beta(b). \Rightarrow \beta(b) \in \operatorname{Ker} j' = \operatorname{Im} i',$$

so $a' \in A'$ exists such that $i'(a') = \beta(b)$.
Since α is an isomorphism $a \in A$ exists s.t. $\alpha(a) = a'$. Then

$$\beta(b) = i' \alpha(a) = \beta\, i(a)$$

Since β is an isomorphism,

$$b = i(a).$$

exactness.

$$c = j(b) = j\, i(a) = 0$$

∎

ADDENDUM to theorem SES $\Rightarrow$ LES

Let

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_\bullet \arrow[r, "i"] \arrow[d, "f"] & \mathcal{B}_\bullet \arrow[r, "j"] \arrow[d, "g"] & \mathcal{C}_\bullet \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathcal{A}_\bullet' \arrow[r, "i'"] & \mathcal{B}_\bullet' \arrow[r, "j'"] & \mathcal{C}_\bullet' \arrow[r] & 0
\end{tikzcd}$$

be two SES of chain complexes and f, g, h chain maps s.t. the diagram above commutes. Then we obtain two LES in homology with maps between them that makes all the squares commutative

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_p(\mathcal{A}_\bullet) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_p(\mathcal{B}_\bullet) \arrow[r, "j_*"] \arrow[d, "g_*"] & H_p(\mathcal{C}_\bullet) \arrow[r, "\partial_*"] \arrow[d, "h_*"] & H_{p-1}(\mathcal{A}_\bullet) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_p(\mathcal{A}_\bullet') \arrow[r, "i_*'"] & H_p(\mathcal{B}_\bullet') \arrow[r, "j_*'"] & H_p(\mathcal{C}_\bullet') \arrow[r, "\partial_*"] & H_{p-1}(\mathcal{A}_\bullet') \arrow[r] & \cdots
\end{tikzcd}$$

Proof of MV, #2

$U \simeq A$ $\quad$ $V \simeq B$ $\quad$ & $\quad$ $U \cap V \simeq A \cap B$

shows deformation retract

We can observe

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A\cap B) \arrow[r] \arrow[d, hook] & S_\bullet(A)\oplus S_\bullet(B) \arrow[r] \arrow[d, hook] & S_\bullet^{\{A,B\}}(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & S_\bullet(U\cap V) \arrow[r] & S_\bullet(U)\oplus S_\bullet(V) \arrow[r] & S_\bullet^{\{U,V\}}(X) \arrow[r] & 0
\end{tikzcd}$$

these two SES sequences induce the following commutative diagram (addendum)

$$\begin{tikzcd}
\cdots \arrow[r] & H_{p+1}^{\{A,B\}}(X) \arrow[r] \arrow[d] & H_p(A\cap B) \arrow[r] \arrow[d, "\cong"] & H_p(A)\oplus H_p(B) \arrow[r] \arrow[d, "\cong"] & H_p^{\{A,B\}}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{p+1}^{\{U,V\}}(X) \arrow[r] \arrow[d, phantom, "\cong"] & H_p(U\cap V) \arrow[r] & H_p(U)\oplus H_p(V) \arrow[r] & H_p^{\{U,V\}}(X) \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \\
 & H_{p+1}(X) & & & H_p(X) &
\end{tikzcd}$$

By the 5-lemma $H_{p+1}^{\{A,B\}}(X) \cong H_{p+1}^{\{U,V\}}(X) \cong H_{p+1}(X)$.