EQUIVALENCE OF SIMPLICIAL & SINGULAR HOMOLOGY

Recall that simplicial homology was defined in terms of a Δ-complex decomposition of X, via a collection of maps

$$\sigma_\alpha : \Delta^p \to X.$$

We then defined the chains to be the free abelian group on the p-simplices, $\Delta_p(X)$. We will now show that if a Δ-complex structure is chosen, then its simplicial homology coincides with the singular homology of the space X.

We will do this by induction on the skeleton $X^{(k)}$ consisting of all simplices of dimension k or less, and so we would like to use a relative version of simplicial homology.

LEMMA 1

For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$\oplus i_{\alpha *} : \bigoplus_\alpha \tilde{H}_p(X_\alpha) \to \tilde{H}_p\left(\bigvee_\alpha X_\alpha\right),$$

provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

Recall

The wedge sum is a one-point union of a family of topological spaces:

$$\bigvee_\alpha (X_\alpha, x_\alpha) = \left(\bigsqcup_\alpha X_\alpha\right) \Big/ \{x_\alpha\}_\alpha$$

quotient of the disjoint union of the X_α's by the equivalence relation that identifies all x_αs with each other and makes no other identifications

Proof

Since (X_α, x_α)'s are good pairs,

$\left(\bigsqcup_\alpha X_\alpha, \{x_\alpha : \alpha \in I\}\right)$ is a good pair,

so

$$H_p\left(\bigsqcup_\alpha X_\alpha, \{x_\alpha : \alpha \in I\}\right) \underset{q_*}{\xrightarrow{\cong}} \tilde{H}_p\left(\bigsqcup_\alpha X_\alpha \Big/ \{x_\alpha\}_\alpha\right) = \tilde{H}_p\left(\bigvee_\alpha X_\alpha\right)$$

Furthermore, for pairs of topological spaces we have

induced by inclusions

$$H_p\left(\bigsqcup_\alpha (A_\alpha, B_\alpha)\right) \cong \bigoplus_\alpha H_p(A_\alpha, B_\alpha).$$

It follows from here that

$$\tilde{H}_p\left(\bigvee_\alpha X_\alpha\right) \cong \bigoplus_\alpha H_p(X_\alpha, x_\alpha)$$

We did an example a while back

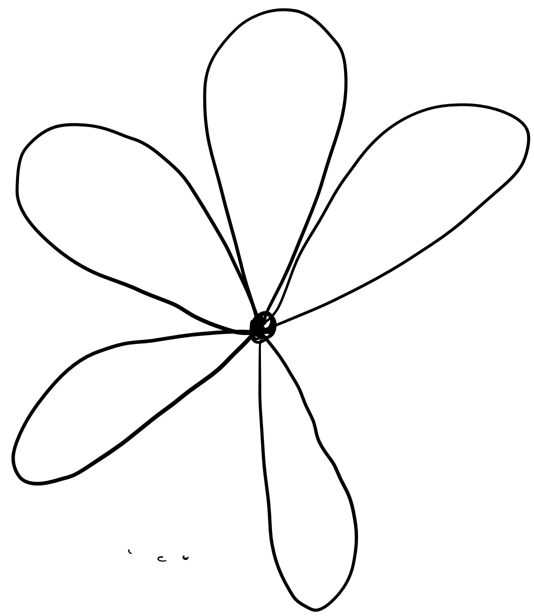
$$\cong \bigoplus_\alpha \tilde{H}_p(X_\alpha)$$

and showed that

$$H_p(X, x_0) \cong \tilde{H}_p(X).$$

EXAMPLE

Let $X = \underbrace{S^n \vee S^n \vee \dots \vee S^n}_{\text{a finite wedge of } h \text{ n-spheres}}$



Then by Lemma 1

$$\tilde{H}_p(\vee S^n) \cong \tilde{H}_p(S^n) \oplus \dots \oplus \tilde{H}_p(S^n)$$

$$= \begin{cases} \bigoplus_h \mathbb{Z} & p = n \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION (RELATIVE SIMPLICIAL HOMOLOGY)

Let X be a Δ-complex & A a Δ-subcomplex.
We define the quotient chain complex

$$\underbrace{\Delta_p(X,A)}_{\text{relative simplicial chains}} = \Delta_p(X) / \Delta_p(A)$$

with ∂ restrictions of simplicial boundary maps. The homology of this chain complex is denoted by $H_p^{\Delta}(X,A)$.