DEFINITION (RELATIVE SIMPLICIAL HOMOLOGY)

Let X be a Δ-complex & A a Δ-subcomplex. We define the quotient chain complex

$$\Delta_p(X,A) = \Delta_p(X) / \Delta_p(A)$$

relative simplicial chains

with ∂ restrictions of simplicial boundary maps. The homology of this chain complex is denoted by $H_p^{\Delta}(X,A)$.

THEOREM

maps that give the Δ-complex structure

the characteristic map of any p-simplex in a Δ-complex decomposition of X may be viewed as a singular p-simplex, hence we have the inclusion chain map i_c

$$\begin{array}{ccccccc}
\cdots \to & \Delta_p(X,A) & \to & \Delta_{p-1}(X,A) & \to & \Delta_{p-2}(X,A) & \to \cdots \\
& \downarrow i_c & & \downarrow i_c & & \downarrow i_c & \\
\cdots \to & S_p(X,A) & \to & S_{p-1}(X,A) & \to & S_{p-2}(X,A) & \to \cdots
\end{array}$$

The induced homomorphism

$$H_p^{\Delta}(X,A) \to H_p(X,A) \text{ is an isomorphism.}$$

Taking $A=\phi$, we obtain the equivalence of absolute singular and simplicial homology.

Proof

(i) special case : $A=\phi$, X is finite-dimensional

We denote by $X^{(k)}$ the k-skeleton of X.

Then we have the following LES

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H_{p+1}^{\Delta}(X^{(k)},X^{(k-1)}) \arrow[r] \arrow[d, "1"] & H_p^{\Delta}(X^{(k-1)}) \arrow[r] \arrow[d, "2"] & H_p^{\Delta}(X^{(k)}) \arrow[r] \arrow[d, "3"] & H_p^{\Delta}(X^{(k)},X^{(k-1)}) \arrow[r] \arrow[d, "4"] & H_{p-1}^{\Delta}(X^{(k-1)}) \arrow[r] \arrow[d, "5"] & \cdots \\
\cdots \arrow[r] & H_{p+1}(X^{(k)},X^{(k-1)}) \arrow[r] & H_p(X^{(k-1)}) \arrow[r] & H_p(X^{(k)}) \arrow[r] & H_p(X^{(k)},X^{(k-1)}) \arrow[r] & H_{p-1}(X^{(k-1)}) \arrow[r] & \cdots
\end{tikzcd}$$

First we show that 1 & 4 are an isomorphism.

Note that $\Delta_p(X^{(k)},X^{(k-1)})=0$ for $p\neq k$

& furthermore, $\Delta_k(X^{(k)},X^{(k-1)})$ is a free abelian group generated by k-simplices in X

Hence, $H_p^{\Delta}(X^{(k)}, X^{(k-1)}) = \begin{cases} 0 & p \neq k \\ \bigoplus_{\substack{\# k\text{-sx} \\ \text{in } X}} \mathbb{Z} & p = k \end{cases}$

To compute $H_p(X^{(k)}, X^{(k-1)})$ note that $X^{(k-1)}$ is a strong deformation retract of its neighborhood in $X^{(k)}$.

(proposition A.5. on page 523 in Hatcher).

So

$$H_p(X^{(k)}, X^{(k-1)}) \xrightarrow{\cong} \tilde{H}_p\left(X^{(k)}/X^{(k-1)}\right).$$

$$X^{(k)}/X^{(k-1)} = \bigvee_{\substack{\# k\text{-sx} \\ \text{in } X}} S^k \Rightarrow$$

$$\tilde{H}_p\left(X^{(k)}/X^{(k-1)}\right) = \tilde{H}_p\left(\bigvee_{\substack{\# k\text{-sx} \\ \text{in } X}} S^k\right) = \begin{cases} 0 & p \neq k \\ \bigoplus_{\substack{\# k\text{-sx} \\ \text{in } X}} \mathbb{Z} & p = k \end{cases}$$

$\Rightarrow H_p^{\Delta}(X^{(k)}, X^{(k-1)}) \to H_p(X^{(k)}, X^{(k-1)})$ is an isomorphism.

$$\begin{tikzcd}[column sep=small]
\to H^{\Delta}_{p+1}(X^{(k)}, X^{(k-1)}) \arrow[r] \arrow[d, "①"', "\cong"] & H^{\Delta}_p(X^{(k-1)}) \arrow[r] \arrow[d, "②"] & H^{\Delta}_p(X^{(k)}) \arrow[r] \arrow[d, "③"] & H^{\Delta}_p(X^{(k)}, X^{(k-1)}) \arrow[r] \arrow[d, "④"', "\cong"] & H^{\Delta}_{p-1}(X^{(k-1)}) \to \arrow[d, "⑤"] \\
\to H_{p+1}(X^{(k)}, X^{(k-1)}) \arrow[r] & H_p(X^{(k-1)}) \arrow[r] & H_p(X^{(k)}) \arrow[r] & H_p(X^{(k)}, X^{(k-1)}) \arrow[r] & H_{p-1}(X^{(k-1)}) \to
\end{tikzcd}$$

So ① & ④ are isomorphisms.

All homology groups of the empty set $X^{(-1)}$ are trivial. Assume that $H^{\Delta}_p(X^{(j)}) \to H_p(X^{(j)})$ are isomorphisms for $j = k-1$ and for all p.

By induction on k we may assume that the second ② and the fifth ⑤ map are isomorphisms as well.

Then ③ is an isomorphism by the five-lemma.

(ii) X infinite dimensional, $A = \phi$

FACT A compact set in X can meet X in only finitely many open simplices of X, ie simplices with their proper faces deleted.

Proof of the fact

If a compact set C intersected infinitely many open simplices, it would contain an infinite sequence of points x_i each lying in a different open simplex. Then

$$U_i = X - \bigcup_{j \neq i} \{x_j\}$$

are open since their preimages under the characteristic maps of all simplices are open. They also form an open cover with no subcover.

Let us show that

① $H_p^\Delta(X) \to H_p(X)$ is surjective

Let $m \in H_p(X)$, $m = [c]$, $c \in S_p(X)$, $\partial c = 0$.

c is a finite linear combination of singular simplices, whose images are compact subsets of X. So the images of simplices are a compact set. Using the fact above, we deduce that these images are in $X^{(k)}$ for k big enough. $\Rightarrow c \in S_p(X^{(k)})$.

By (i) we know that $H_p^{\Delta}(X^{(k)}) \to H_p(X^{(k)})$ is an isomorphism, so $v \in \Delta_p(X^{(k)})$ exists, $\partial v = 0$ such that $[v]$ gets mapped to $[c] \in H_p(X^{(k)})$.

Now $v \in \Delta_p(X^{(k)}) \subset \Delta_p(X)$ &

$[v]$ is mapped to $[c] \in H_p(X)$.

$$\begin{tikzcd}
H_p^{\Delta}(X^{(k)}) \arrow[r] \arrow[d] & H_p(X^{(k)}) \arrow[d] \\
H_p^{\Delta}(X) \arrow[r] & H_p(X)
\end{tikzcd}$$

② $H_p^{\Delta}(X) \to H_p(X)$ is injective

Let $c \in \Delta_p(X)$, $[c]$ is in the kernel of the above homomorphism. This means that c is a boundary of some singular $(p+1)$-chain $b \in S_{p+1}(X)$.

Both chains, c and b, lie in $X^{(k)}$ for a large enough k (the argument is the same as before). This means that c represents an element in the kernel of

$$\begin{aligned}
H_p^{\Delta}(X^{(k)}) &\xrightarrow{\overset{(i)}{\cong}} H_p(X^{(k)}) \\
[c] &\longmapsto 0
\end{aligned}$$

$\Rightarrow [c] = 0 \in H_p^\Delta(X^{(k)}) \Rightarrow [c] = 0$

in $H_p^\Delta(K)$.

(iii) X, A general, $A \subset X$. Then LES for (X, A) yields

$$\begin{tikzcd}
\to H_{p+1}^\Delta(X,A) \arrow[r] \arrow[d] & H_p^\Delta(A) \arrow[r] \arrow[d, "\cong (ii)"] & H_p^\Delta(X) \arrow[r] \arrow[d, "\cong (ii)"] & H_p^\Delta(X,A) \arrow[r] \arrow[d] & H_{p-1}^\Delta(A) \to \arrow[d, "\cong (ii)"] \\
\to H_{p+1}(X,A) \arrow[r] & H_p(A) \arrow[r] & H_p(X) \arrow[r] & H_p(X,A) \arrow[r] & H_{p-1}(A) \to
\end{tikzcd}$$

By the 5-lemma $H_p^\Delta(X,A) \to H_p(X,A)$ is also an isomorphism.