$\Rightarrow f(x) \neq x, -x \quad \forall x$. this is a contradiction with the previous corollary.

APPLICATION

$n=2$, S^2 = Earth's surface

$\vec{v}(x)$ = velocity of the wind at $x \in$ Earth

$\Rightarrow$ at any given moment, $\exists$ a point $x \in$ Earth where the wind does not blow.

Exercise: Show that S^{2k-1} (odd dim sphere) does have a nowhere vanishing vector field. (HW exercise)

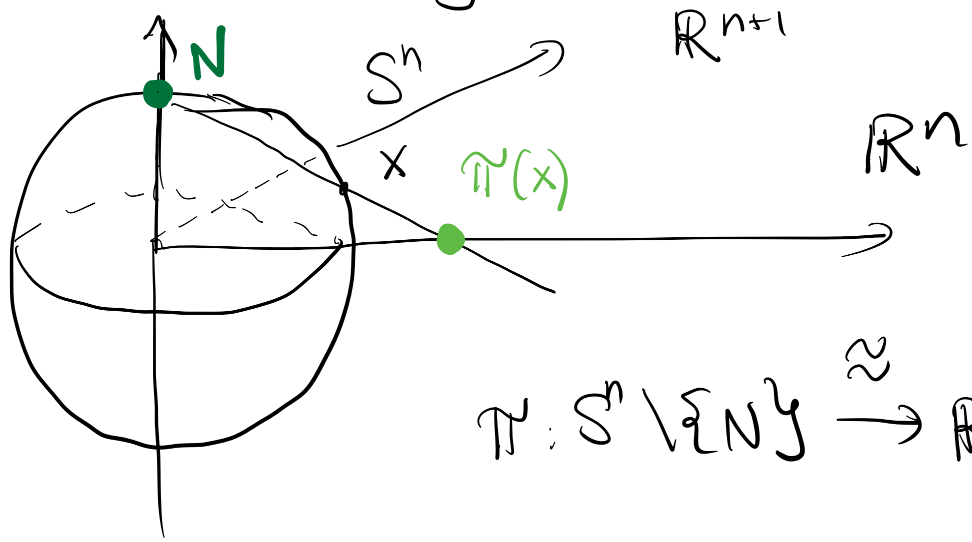
CALCULATION OF DEGREES

Consider maps $S^n \to S^n$ defined as follows.

$S^n \approx \mathbb{R}^n \cup \{\infty\}$ (one-point compactification)

open sets of the original space + nbhd of $\{\infty\}$ [complement of a comp-set $\cup \{\infty\}$)

Recall the stereographic projection



$$\pi : S^n \setminus \{N\} \xrightarrow{\approx} \mathbb{R}^n$$

π extends to a homeomorphism that we denote by $\hat{\pi} : S^n \to \mathbb{R}^n \cup \{\infty\}$ (HW exercise)

Now fix a non-singular $n \times n$ matrix A. ($\det A \neq 0$). View A as a homeomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A \cdot x$. (invertible since A is non-singular).

A extends to a homeo.

$$\mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$$

(HW exercise)

Denote this extension by

$$\hat{A} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}.$$

Remark: If $\det A = 0$, A does not extend to $\hat{A} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$

(there are points in the kernel, so $\|x\| \to \infty$ $\|Ax\| \to 0$, but also points with $\|x\| \to \infty$ $\|Ax\| \not\to 0$)

PROPOSITION

$\deg \hat{A} = \operatorname{sgn} \det A$. $\left(\operatorname{sgn} = \begin{cases} +1 \\ -1 \end{cases} \right)$

Proof

Observation: If the statement holds for A' & A'', then it holds also for $A' \cdot A''$. This is true because

$\widehat{A' \cdot A''} = \hat{A}' \circ \hat{A}''$ (check) and

$\deg(f' \circ f'') = \deg f' \cdot \deg f''$:

$\deg(\widehat{A' \cdot A''}) = \deg(\hat{A}' \circ \hat{A}'') = \deg \hat{A}' \cdot \deg \hat{A}''$

$= \operatorname{sgn} \det A' \cdot \operatorname{sgn} \det A'' = \operatorname{sgn} \det A' \cdot \det A'' = \operatorname{sgn} \det A' \cdot A''$

Every non-singular matrix A can

be written as a product

$A = E_1 \cdot \ldots \cdot E_p$ of elementary matrices

elementary matrices

E_i, where each E_i is of one of the following types:

(I) $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & \lambda & & \\ & 0 & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$, $\lambda \neq 0$ multiply a row by λ

(II) $\begin{array}{c|c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & \cdots & 0 & \\ & & & \vdots & \ddots & \vdots & \\ & & & \lambda & \cdots & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \leftarrow j \\ \leftarrow k \end{array} \\ \quad\quad\quad\quad j \quad\quad k & \end{array}$, $\lambda \in \mathbb{R}$ multiply from the right: take ~~kth~~ row & add λ jth row

(III) $\begin{array}{c|c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & \vdots & 1 & \vdots & \\ & & & 1 & \cdots & 0 & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \leftarrow j \\ \leftarrow k \end{array} \\ \quad\quad\quad\quad j \quad\quad k & \end{array}$ take id and interchange rows j & k

It's enough to check our proposition holds for each of these types.

Case I. $\hat{E} \simeq$ either to $\hat{id}$ or to $\widehat{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}$,

homotopic as map $S^n \to S^n$

depending on the sign of λ.

Case II $\hat{E} \simeq \hat{id}$

Case III E performs a reflection wrt. some hyperplane. → equidistant to vectors j & k

$$\hat{E} \simeq \widehat{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}}$$

reflection wrt. $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$

In all 3 cases we get $\deg(\hat{E}) = \operatorname{sgn} \det(E)$ & so the proof follows.

Remark In the proof it is crucial to homotope E to another matrix by a path E_t of NON-SINGULAR matrices. Otherwise, E_t won't extend to $\hat{E}_t$.

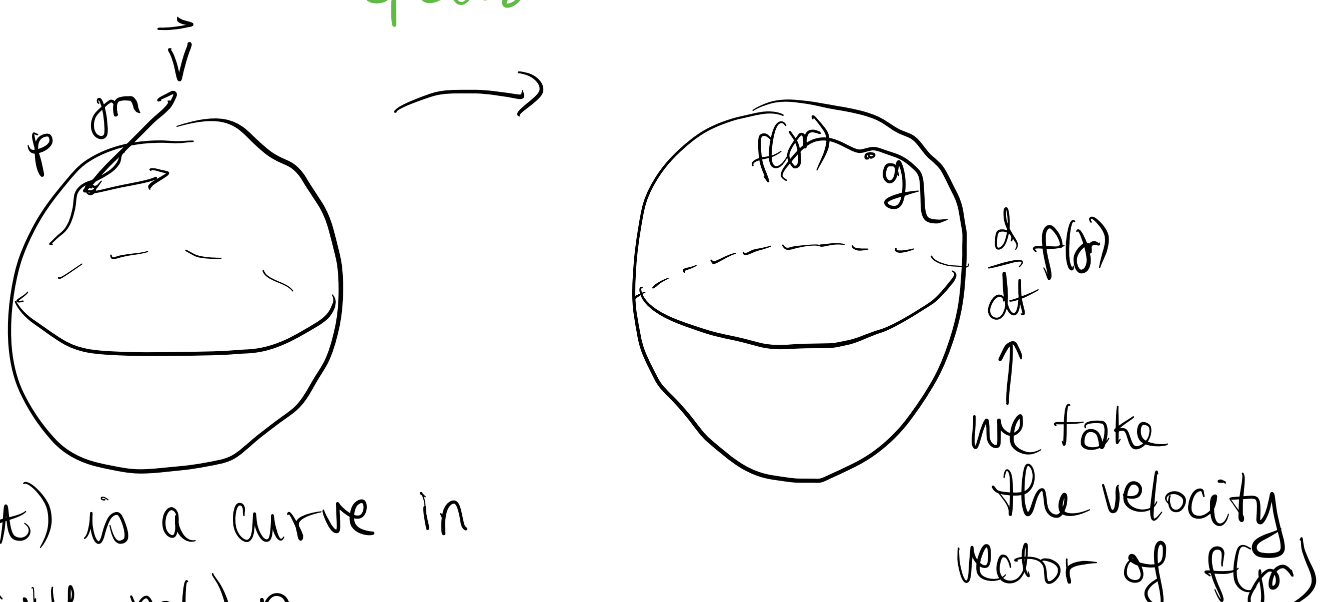
See Introduction to Smooth Manifolds by J.M. Lee.

Let $f: S^n \to S^n$ be a smooth map.

Let $p \in S^n$, $q := f(p) \in S^n$. Then

$Df_p : T_p(S^n) \to T_q(S^n)$

different linear spaces



$\gamma(t)$ is a curve in M with $\gamma(0) = p$

Remark: Taking $\det(Df_p)$ doesn't make sense since it depends on the choice of bases.

We'll define $\varepsilon_p(f)$ as follows:

choose $\sigma \in SO(n+1)$ (orthogonal matrix with $\det = +1$) st. $\sigma(q) = p$.

Consider $\sigma \circ f : S^n \to S^n$, $\sigma \circ f(p) = p$

Consider

$D(\sigma \circ f)_p : T_p(S^n) \to T_p(S^n)$

Put $\quad \varepsilon_p(f) := \operatorname{sgn} \det \left(D(\sigma \circ f)_p \right)$

this ε_p can be $+1, 0$ or -1

$\varepsilon_p(f)$ does not depend on σ. Indeed, if $\sigma'(q) = p$ is another such map,

$$\sigma' \circ f = \underbrace{(\sigma' \circ \sigma^{-1})}_{\det = +1} \circ (\sigma \circ f)$$

$$\Rightarrow \det D(\sigma' \circ f) = \det D(\sigma \circ f).$$

PROPOSITION

Let $f: S^n \to S^n$ be a smooth map. Assume that $q \in S^n$ is a regular value of f & $f^{-1}(q) = \{p\}$. Then $\deg(f) = \varepsilon_p(f)$.

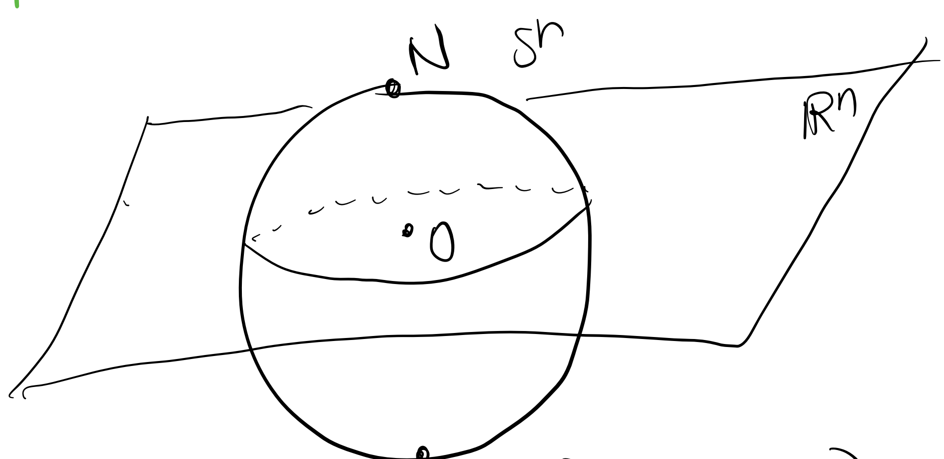
Recall.

X, Y smooth manifolds, $f: X \to Y$. q is called a regular value of f if <u>either</u> $f^{-1}(q) = \emptyset$ or $\forall x \in f^{-1}(q)$ the map $Df_x : T_x(X) \to T_q(Y)$ is surjective (in our case an isomorphism)

(Note that $\varepsilon_p(f) = \pm 1$, but not 0)

Proof

Identify S^n with $\mathbb{R}^n \cup \{\infty\}$ via the stereographical projection.



$S = p = q$ (south pole)

WLOG assume $p = q = 0$. this is possible since we can always compose f with a suitable $\gamma \in SO(n+1)$, and $SO(n+1)$ is path connected (exercise in HW) hence,

$$f \simeq \sigma_2 \circ (\gamma \circ f) \circ \sigma_1$$

γ maps q to p
σ_1 maps 0 to p
σ_2 maps p to 0

$\gamma, \sigma_1, \sigma_2 \in SO(n+1)$

$$\Rightarrow \deg f = \deg \sigma_2 \circ (\gamma \circ f) \circ \sigma_1$$

Also $\varepsilon_0(\sigma_2 \circ (\gamma \circ f) \circ \sigma_1) = \operatorname{sgn} \det D(\sigma_2 \circ (\gamma \circ f) \circ \sigma_1)_0 =$

chain rule

$= \operatorname{sgn} \det (\sigma_2 \cdot D(\gamma \circ f)_p \cdot \sigma_1) = \operatorname{sgn} \det \sigma_2 \cdot \det D(\gamma \circ f)_p \cdot \det \sigma_1 =$

$= \operatorname{sgn} D(\gamma \circ f)_p = \varepsilon_p(f)$

Step 1 Assume $Df_{(0)} = id$.

By Taylor's theorem from analysis we can write

$$f(x) = x + g(x) \text{ for } \|x\| \leq \delta, \text{ where } \|g(x)\| = o(|x|)$$

$$\frac{\|g(x)\|}{\|x\|} \to 0$$

$\Rightarrow$ For $\varepsilon > 0$ small enough

$$\frac{\|g(x)\|}{\|f(x)\|} = \frac{\|g(x)\|}{\|x + g(x)\|} \leq \frac{\frac{\|g(x)\|}{\|x\|}}{\left\|1 - \frac{\|g(x)\|}{\|x\|}\right\|} < \frac{1}{100} \quad \forall\, 0 < |x| \leq 2\varepsilon$$

$\to 0$

$\to 0$

$f(x) \neq 0$ for $x \neq 0$ by assumption

Define a homotopy $F: S^n \times I \to S^n$ as follows:

$$F(x,t) := \begin{cases} f(x) & 2\varepsilon \leq |x| \\ f(x) - t\left(2 - \frac{|x|}{\varepsilon}\right)g(x) & \varepsilon \leq |x| \leq 2\varepsilon \\ f(x) - tg(x) & |x| \leq \varepsilon \end{cases}$$

interpolation

F is well-defined and continuous.

$F(x,0) = f(x)$. Put $f_1(x) := F(x,1)$.

Note that $f_1(x) = x$ for all $|x| \leq \varepsilon$.

Claim: $\forall x \neq 0$, $f_1(x) \neq 0$.

Proof of claim: For $|x| \geq 2\varepsilon$, $f_1(x) = f(x)$.

assumption: f admits g only once at p

If $\varepsilon \leq |x| \leq 2\varepsilon$, then

homotopy formula

$$f_1(x) = 0 \iff f(x) = \frac{2\varepsilon - |x|}{\varepsilon} g(x)$$