COROLLARY

Let $f: S^n \to S^n$ be a smooth map and let $p \in S^n$ be a regular value. Assume that $f^{-1}(p) = \{a_1, \dots, a_k\}$.

Then $$\deg f = \sum_{j=1}^{k} \varepsilon_{a_j}(f).$$

local degree of f at a_j

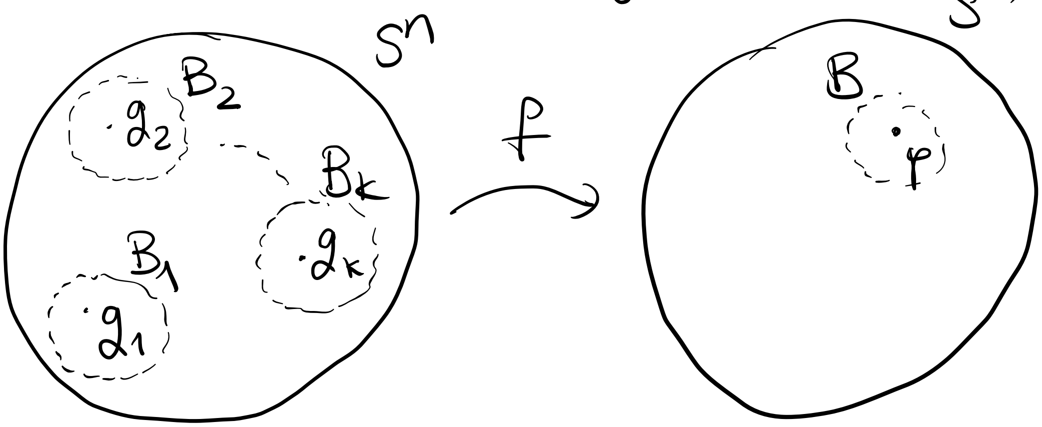
If $f^{-1}(p) = \phi$ (i.e. f is not surjective), then $\deg f = 0$.

(Note: this result is independent of homology theory as long as the coefficient group is $\mathbb{Z}$)

Proof

Assume first that $f^{-1}(p) \neq \phi$. By the implicit function theorem there exists an open ball $B \subset S^n$ around p s.t.

$f^{-1}(B) = \bigsqcup_{j=1}^{k} B_j$, where B_j is an open ball around q_j which is mapped by f diffeomorphically onto B



Consider a homotopy $F: S^n \times I \to S^n$ with $F(x,0) = x \quad \forall x \in S^n$ and such that

$$\Phi(x) := F(x,1) \text{ satisfies the}$$

following

antipodal point to p

① $\Phi(B) = S^n \setminus \{-p\}$

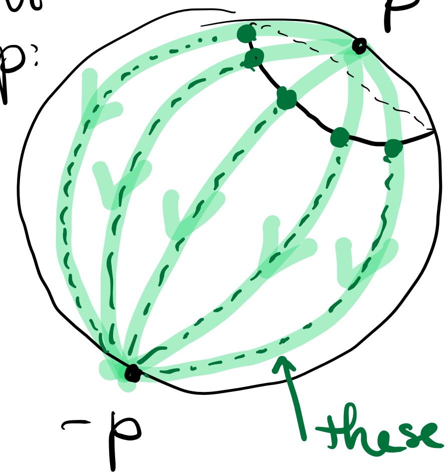
② $\Phi(S^n \setminus B) = \{-p\}$

③ $\Phi(p) = p$

④ Φ is smooth, and $D\Phi_p : T_p(S^n) \to T_p(S^n)$ is the identity

Exercise: Such a map exists.

Idea for the map:



Consider $F(f(x), t)$. This gives us a homotopy between f and $\phi \circ f$ $\Rightarrow$

$\deg(f) = \deg(\phi \circ f)$.

For $\deg(\phi \circ f)$ we can apply our previous theorem.

$$\Rightarrow \deg f = \deg(\phi \circ f) = \sum_{i=1}^{K} \deg(\phi \circ f_i) =$$

$$= \sum_{i=1}^{k} \varepsilon_{q_i}(\phi \circ f) = \sum_{i=1}^{k} \varepsilon_{q_i}(f)$$

same number (compute the determinant)

Finally, we must check what happens if $p \notin \mathrm{Im}(f)$? In this case

$$\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend right, "f"'] & S^n \setminus \{p\} \arrow[r, hook] & S^n \end{tikzcd}$$

the map f factors as in the diagram above.

Also, $S^n \setminus \{p\} \approx \mathbb{R}^n \Rightarrow \deg f = 0$.

EXAMPLES

$S^1 \subset \mathbb{C}$ unit circle.

$$f : S^1 \longrightarrow S^1, \; f(z) = z^k, \; k \in \mathbb{Z}.$$

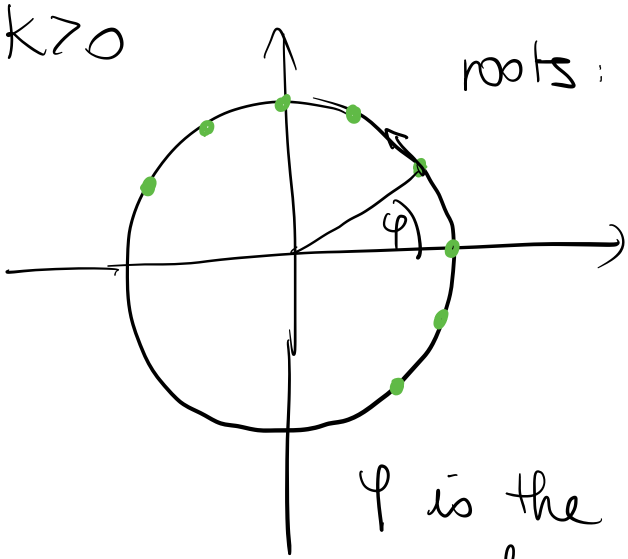
Then $\deg(f) = k$.

$f(z) = z^k \qquad S^1 \subset \mathbb{C}.$

$k = 0 \qquad f(z) = 1$

$\deg f = 0$ since the map is not surjective

$k > 0$



roots: $e^{j\frac{2\pi i}{k}}$

φ is the angle

locally f is:

$$\varphi \mapsto k\varphi$$

$$D(k\varphi) = k$$

$$\deg f_i = \operatorname{sgn} D(k\varphi) = +1$$

$$\Rightarrow \deg f = \sum \deg f_i = k$$

$k < 0$

The argument is similar.

$$D(k\varphi) = k$$

$$\deg f_i = -1$$

$$\Rightarrow \deg f = \sum_{i=1}^{|k|} \deg f_i = -k.$$

APPLICATION (THE FUNDAMENTAL THEOREM OF ALGEBRA)

Let $p(x)$ be a non-constant polynomial (with $\mathbb{C}$-coefficients). Then $p : \mathbb{C} \to \mathbb{C}$ is surjective.

Proof

If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_n \neq 0$, then

$$|p(z)| = |a_n||z|^n \left| 1 + \frac{a_{n-1}}{|a_n||z|} + \dots + \frac{a_0}{|a_n||z|^n} \right| > \frac{1}{2}$$

for large $|z|$

$$\Rightarrow \lim_{|z| \to \infty} |p(z)| = \infty.$$

Conclusion: p extends to a map $p : S^2 \to S^2$. This extension is smooth.

1-point compactification of $\mathbb{C}$

(Milnor: Topology from differentiable viewpoint)

Critical points of p: points $z \in \mathbb{C} \cup \{\infty\}$ where $D_z p$ is not surjective.

By standard complex analysis this happens $\iff$ $p'(z) = 0$.

complex derivative

Since p is not constant 0 $\exists w \in \mathbb{C}$ which is a regular value of p <u>and</u> $p^{-1}(w) \neq \phi$.

Write $p(z) = u(z) + iv(z)$.

$$Dp_z = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

$z = x + iy$

by Cauchy-Riemann equations

$$\partial_x u = \partial_y v$$

$$\partial_x v = -\partial_y u$$

(from complex analysis).

$\Rightarrow \det Dp_z = (\partial_x u)^2 + (\partial_y u)^2 > 0 \quad \forall$ reg. value z

$\Rightarrow$ local degree of p at every $z \in p^{-1}(w)$ is strictly positive. $\Rightarrow \deg p > 0$.

$\Rightarrow \deg p \neq 0$. this means that $p : S^2 \to S^2$ is surjective. Let now $w \in \mathbb{C} \subset S^2$. Since $p(\infty) = \infty$, we have $\infty \notin p^{-1}(w)$.

$\Rightarrow \exists z \in S^2 \setminus \{\infty\} = \mathbb{C}$ so that $p(z) = w$.