TOPOLOGY ON K

A set $A \subset K$ is open (or closed) iff $f_\sigma^{-1}(A)$ is open (or closed) in B_σ^n for every characteristic map f_σ.

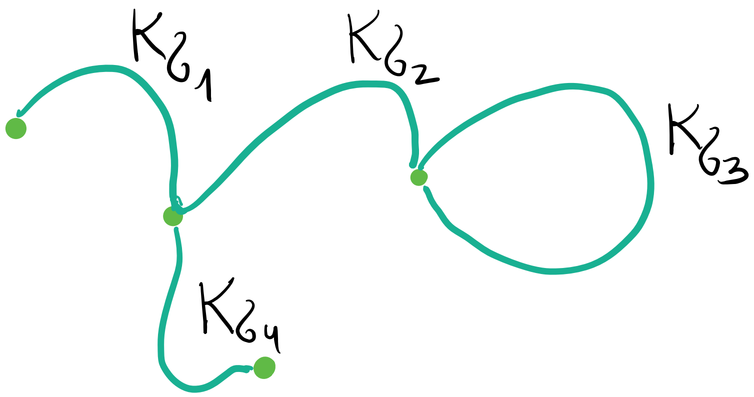
A **SUBCOMPLEX** of a CW complex X is a subspace $A \subset X$ which is a union of cells of X such that the closure of each cell in A is contained in A, ie. A is a CW complex.

EXAMPLES

① 0-dim CW-complex = space with discrete topology.

② 1-dim CW-complex - graph

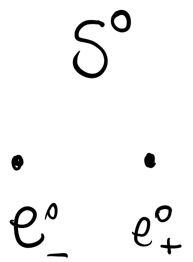
($K^{(0)}$ = vertices, $K^{(1)}$ = edges + vertices)

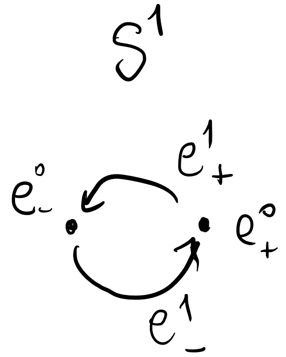


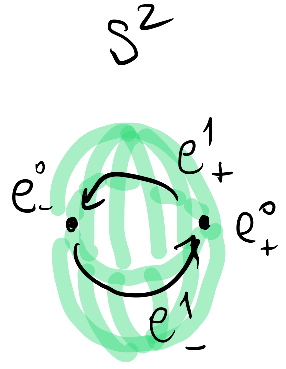
③ Any simplicial complex is a CW-complex.

④ The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$.

Alternative:







$$e^n_\pm = \{ (x_0, \dots, x_n) \mid x_n > 0 \}$$

$$\cong \mathring{B}^n$$

$$S^n = \underset{\nearrow}{e^n_+} \cup \underset{\nwarrow}{e^n_-} \cup S^{n-1}$$

upper hemisphere

lower hemisphere

Advantage: S^n is a subcomplex of S^{n+1}!

We also define

$$\bigcup_{n=0}^{\infty} S^n \overset{\text{def}}{=} S^\infty \leftarrow$$ in each dimension we have two cells

BUILDING SPACES FROM CW-COMPLEXES

① PRODUCT

$(X, \mathcal{E}), (Y, \mathcal{F})$ CW-complexes

$\mathcal{E}$: cells of X; $\mathcal{F}$: cells of $\mathcal{F}$

$\Rightarrow (X \times Y, \mathcal{E} \times \mathcal{F})$ is also a CW-complex,

where

$$(\mathcal{E} \times \mathcal{F})_n = \{ e \times e' \mid e \in \mathcal{E}_k, e' \in \mathcal{F}_s, k+s=n \}$$

and the characteristic maps

$$f_e : B^k \to \bar{e} \subset X \quad \& \quad f_{e'} : B^s \to \bar{e}' \subset Y$$

induce $f_e \times f_{e'} : B^k \times B^s \to \overline{e \times e'} \subset X \times Y$.

Complication : For completely general CW complexes, $X \times Y$ as a cell complex sometimes has finer topology than the product topology, though the two coincide if either X or Y has finitely many cells.

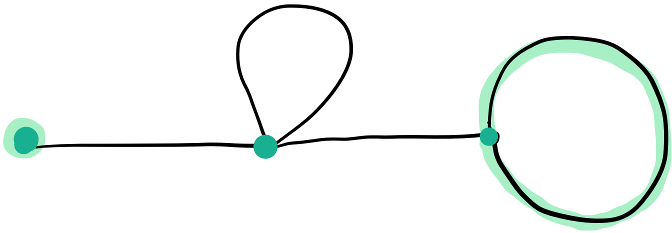
② QUOTIENT

X a CW-complex & A a subcomplex of A. $\Rightarrow$ X/A has a natural CW decomposition as a CW-complex.

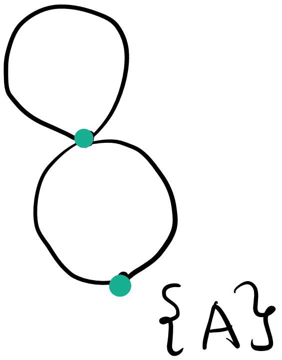
The cells of X/A are the cells of $X-A$ plus one new 0-cell, the image of A in X/A.

The characteristic maps for cells are characteristic maps of cells from ε, composed with the quotient map $X \to X/A$.

X



X/A



Other constructions: suspension, join, wedge sum, smash product (see Hatcher)

THEOREM

Let A be a subcomplex of X, U an open neighborhood of A in X. then an open neighborhood V exists, $A \subset V$, $\overline{V} \subset U$ such that A is a strong deformation retract of V. (Hatcher, A.5. page 523).

CELLULAR HOMOLOGY

Cellular homology is a very efficient tool for computing the homology groups of CW-complexes, based on degree calculations.

Let K be a CW-complex. Consider

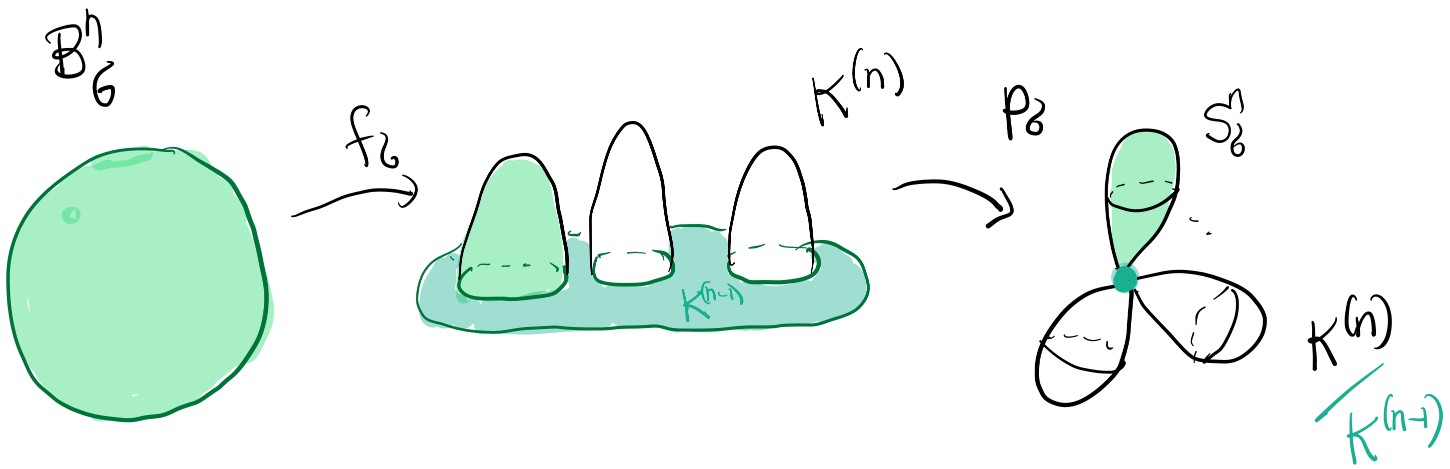
$K^{(n)}/K^{(n-1)}$ (if $n=0$, we take $K^{(0)}$)

(This space has a base point that we denote by $*$. We have $K^{(n)}/K^{(n-1)} \approx \bigvee_{\delta\in I_n} S^n$.

$$B^n_\delta \xrightarrow{f_\delta} K^{(n)} \longrightarrow \frac{K^{(n)}}{K^{(n-1)}} \xrightarrow{\pi_\delta} S^n := S^n_\delta$$

this composition of maps sends ∂B^n_δ to $* \in S^n$. This composition is injective on $\mathring{B}^n_\delta$

projection to the δth sphere in the bouquet $\bigvee_{\delta\in I_n} S^n$



Let $Z^{(n)} := \coprod_{\sigma \in I_n} B^n_\sigma$, and let $f : Z^{(n)} \to K^{(n)}$ be the map coming from all the characteristic maps f_σ. We have

$f(\partial Z^{(n)}) \subset K^{(n-1)}$

Also, note $H_n(Z^{(n)}, \partial Z^{(n)}) \cong \bigoplus_{\sigma \in I_n} H_n(B^n_\sigma, \partial B^n_\sigma)$.

LEMMA

① $\bigoplus_{\sigma \in I_n} (f_\sigma)_* : \bigoplus_{\sigma \in I_n} H_n(B^n_\sigma, \partial B^n_\sigma) \to H_n(K^{(n)}, K^{(n-1)})$

is an isomorphism.

③ Check for 0.

② $H_p(K^{(n)}, K^{(n-1)}) = 0 \quad \forall p \neq n$.

① + ② can be rephrased as

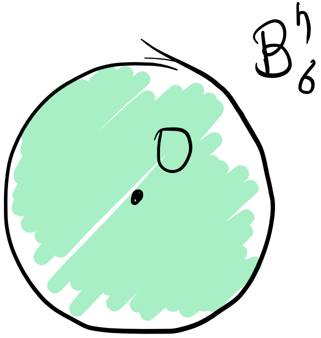
$$\bigoplus_{\sigma\in I_n} (f_\sigma)_* : \bigoplus_{\sigma\in I_n} H_p(B^n_\sigma, \partial B^n_\sigma) \to H_p(K^{(n)}, K^{(n-1)})$$

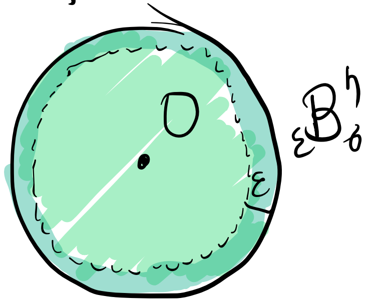
is an isomorphism for all p.

(because $H_p(B^n_\sigma, \partial B^n_\sigma) = 0 \ \forall p \neq n$.)

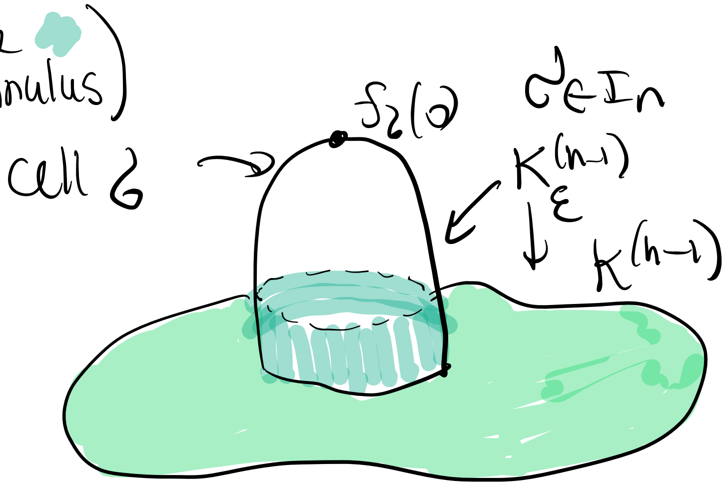
Proof

$$\begin{tikzcd}
H_p(Z^n, \partial Z^n) \arrow[r, "f_*"] \arrow[d, "\cong"', "\text{induced by inclusion}"] & H_p(K^{(n)}, K^{(n-1)}) \arrow[d, "\cong", "\text{(homotopy invariance + 5-lemma)}"'] \\
H_p\Big(\bigsqcup_{\sigma\in I_n}(B^n_\sigma, B^n_\sigma \setminus \{0\})\Big) & H_p(K^{(n)}, K^{(n)} \setminus \cup f_\sigma(0)) \\
H_p\Big(\bigsqcup_{\sigma\in I_n}({}_\varepsilon\mathring{B}^n_\sigma, {}_\varepsilon\mathring{B}^n_\sigma \setminus \{0\})\Big) \arrow[u, "\cong", "\text{induced by inclusion}"'] \arrow[r, "\cong"', "\text{induced by a homeomorphism coming from f}"] & H_p\Big(K^{(n)} \setminus K^{(n-1)}_\varepsilon, K^{(n)} \setminus \big(K^{(n-1)}_\varepsilon \cup \bigcup_{\sigma\in I_n} f_\sigma(0)\big)\Big) \arrow[u, "\cong"', "\text{by excision}"]
\end{tikzcd}$$





$K^{(n-1)}_{\varepsilon} := K^{(n-1)} \cup f_{\sigma}$ (the 🟩 annulus)



$\Longrightarrow f_*$ is an isomorphism.

LEMMA

The following diagram commutes:

$$\begin{tikzcd}
H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_*"] & \tilde{H}_{n-1}(K^{(n-1)}) \\
\bigoplus\limits_{\sigma \in I_n} H_n(B^n_\sigma, \partial B^n_\sigma) \arrow[u, "\bigoplus\limits_{\sigma \in I_n} (f_\sigma)_*"] \arrow[r, "\oplus \partial_*"] & \bigoplus\limits_{\sigma \in I_n} \tilde{H}_{n-1}(\partial B^n_\sigma) \arrow[u, "\bigoplus\limits_{\sigma \in I_n} (f_{\partial\sigma})_*"']
\end{tikzcd}$$

connecting homomorphism

Proof

The LES for homology of pairs is natural with respect to maps.

Consider the LES of $(K^{(n)}, K^{(n-1)})$:

$$\to H_{p+1}(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_p(K^{(n-1)}) \xrightarrow{i_*} H_p(K^{(n)}) \to H_p(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*}$$

- If $p \neq n$, then i_* is surjective.
- if $p \neq n-1$, then i_* is injective
- if $p \neq n, n-1$, then i_* is an isomorphism

Fix a positive index $p > 0$. By induction on n it is easy to show that $H_p(K^{(n)}) = 0$ $\forall p > n$.

Basis: $n = 0$, $H_p(K^{(0)}) = 0$ $\forall j > 0$.

Similarly, it is easy to show that

for $p = n$ we have the exact sequence

(*)

$$\begin{tikzcd}
 & & & 0\partial \arrow[d] & & \\
0 \arrow[r] & H_n(K^{(n)}) \arrow[r, "j_n"] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_n"] \arrow[dr, "\beta_n := j_{n-1} \circ \partial_n"'] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[d, "j_{n-1}"'] & H_{n-1}(K^{(n)}) \arrow[r] & 0 \\
 & & & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[d, "\partial_{n-1}"'] & & \\
 & & & H_{n-2}(K^{(n-2)}) & &
\end{tikzcd}$$