

Consider the LES of $(K^{(n)}, K^{(n-1)})$:

$$\rightarrow H_{p+1}(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_p(K^{(n-1)}) \xrightarrow{i_*} H_p(K^{(n)}) \rightarrow H_p(K^{(n)}, K^{(n-1)}) \rightarrow \dots$$

- If $p \neq n$, then i_* is surjective.
- If $p \neq n-1$, then i_* is injective
- If $p \neq n, n-1$, then i_* is an isomorphism

Fix a positive index $p > 0$. By induction on n it is easy to show that $H_p(K^{(n)}) = 0 \quad \forall p > n$.

Basis: $n=0$, $H_p(K^{(0)}) = 0 \quad \forall p > 0$.

Similarly, it is easy to show that

for $p=n$ we have the exact sequence

$$\begin{array}{ccccccc}
 0 & \rightarrow & H_n(K^{(n)}) & \xrightarrow{j_n} & H_n(K^{(n)}, K^{(n-1)}) & \xrightarrow{\partial_n} & H_{n-1}(K^{(n-1)}) & \rightarrow & H_{n-1}(K^{(n)}) & \rightarrow & 0 & (*) \\
 & & & & & & \downarrow \partial_n & & & & & \\
 & & & & & & H_{n-1}(K^{(n-1)}) & & & & & \\
 & & & & & & \downarrow j_{n-1} & & & & & \\
 & & & & & & H_{n-1}(K^{(n-1)}, K^{(n-2)}) & & & & & \\
 & & & & & & \downarrow \partial_{n-1} & & & & & \\
 & & & & & & H_{n-2}(K^{(n-2)}) & & & & &
 \end{array}$$

$\beta_n = j_{n-1} \circ \partial_n$

For $n+1$ we get

$$\begin{array}{ccccccc}
 & & & & 0 & & (**) \\
 & & & & \downarrow & & \\
 0 \rightarrow & H_{n+1}(K^{(n+1)}) & \xrightarrow{j_{n+1}} & H_{n+1}(K^{(n+1)}, K^{(n)}) & \xrightarrow{\partial_{n+1}} & H_n(K^{(n)}) & \xrightarrow{j_n} H_n(K^{(n+1)}) \rightarrow 0 \\
 & & \searrow \beta_{n+1} & & \downarrow j_n & & \\
 & & & & H_n(K^{(n)}, K^{(n-1)}) & & \\
 & & & & \downarrow \partial_n & & \\
 & & & & H_{n-1}(K^{(n-1)}) & &
 \end{array}$$

CLAIM $\beta_n \circ \beta_{n+1} = 0$.

Proof $\beta_n \circ \beta_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$.

0 since the above sequence is exact

□

CLAIM $\ker \beta_n = \ker \partial_n = \text{Im } j_n$

$\text{Im } \beta_{n+1} = j_n(\text{Im } \partial_{n+1})$.

Proof

j_{n-1} is injective, $\ker \beta_n = \ker \partial_n$.

$\ker \partial_n = \text{Im } j_n$ because the sequence $(*)$ is exact.

Second equality: exercise (diagram chasing).

Conclusion

$$\begin{array}{ccc} H_n(K^{(n)}) & \xrightarrow[\cong]{j_n} & \ker \beta_n \\ \cup & & \\ \text{Im}(\partial_{n+1}) & \xrightarrow[\cong]{j_n} & \text{Im} \beta_{n+1} \end{array} \left. \vphantom{\begin{array}{ccc} H_n(K^{(n)}) & \xrightarrow[\cong]{j_n} & \ker \beta_n \\ \cup & & \\ \text{Im}(\partial_{n+1}) & \xrightarrow[\cong]{j_n} & \text{Im} \beta_{n+1} \end{array}} \right\}$$

$$\Rightarrow H_n(K^{(n+1)}) \cong \text{coker}(\partial_{n+1}) \xrightarrow[\cong]{\beta_n} \frac{\ker \beta_n}{\text{Im} \beta_{n+1}}$$

Proof

In $(**)$ i_n is a surjection

$$H_n(K^{(n+1)}) \cong H_n(K^{(n)}) \xrightarrow[\cong]{\ker i_n} H_n(K^{(n)}) \xrightarrow[\cong]{\text{Im } \partial_{n+1}} \text{coker}(\partial_{n+1})$$

$$\cong \text{coker}(\partial_{n+1})$$

CLAIM In

$$H_n(K^{(n+1)}) \xrightarrow{i_*} H_n(K^{(n+2)}) \xrightarrow{i_*} H_n(K^{(n+3)}) \rightarrow \dots$$

all maps are isomorphisms.

Proof

Let $i \geq n$. then

$$\underbrace{H_{n+1}(K^{(i+1)}, K^{(i)})}_{\cong 0} \rightarrow H_n(K^{(i)}) \rightarrow H_n(K^{(i+1)}) \rightarrow \underbrace{H_n(K^{(i+1)}, K^{(i)})}_{\cong 0} \quad \square$$

COROLLARY

If K is finite dimensional ($\exists n_0$ s.t.

$K^{(n_0)} = K^{(n_0+1)} = \dots = K^{(n_0+i)} = \dots = K$), then

$$H_n(K) \cong H_n(K^{(n+1)}) = \ker \beta_n / \text{Im } \beta_{n+1} \dots H_n(K^{(n_0)}) = H_n(K)$$

DEFINITION

Define a chain complex

$$\begin{aligned} \rightarrow H_{n+1}(K^{(n+1)}, K^{(n)}) \xrightarrow{\beta_{n+1}} H_n(K^{(n)}, K^{(n-1)}) \xrightarrow{\beta_n} H_{n-1}(K^{(n-1)}, K^{(n-2)}) \\ \dots \xrightarrow{\beta_2} H_1(K^{(1)}, K^{(0)}) \xrightarrow{\beta_1} H_0(K^{(0)}) \rightarrow 0 \end{aligned}$$

In this sequence β_1 is the connecting homomorphism from LES of $(K^{(1)}, K^{(0)})$.

We denote the n -homology group of this complex by $H_n^{CW}(K)$.

COROLLARY

If K is finite dimensional, then

$$H_n^{CW}(K) \cong H_n(K).$$

Remark: This statement holds for any CW-complex K , but we will not prove it here.

Since $H_p(K^{(p)}, K^{(p-1)})$ is isomorphic to $\bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma$, we will actually want to work with a chain complex with chain groups

$$C_p^{CW}(K) := \bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma.$$

DEFINITION

Define two homomorphisms:

$$C_n^{CW}(K) \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow{\Phi} \end{array} H_n(K^{(n)}, K^{(n-1)})$$

Recall that $(B_G^n, \partial B_G^n) \xrightarrow{f_G} (K^{(n)}, K^{(n-1)})$

Denote by $[B_G^n]$ the generator of $H_n(B_G^n, \partial B_G^n)$.

$$\Psi \left(\sum_{\sigma} n_{\sigma} \cdot \sigma \right) = \sum_{\sigma} n_{\sigma} (f_{\sigma})_* ([B_G^n])$$

To define Φ , let $\phi_n: H_n(S^n, *) \rightarrow \mathbb{Z}$