Since $H_p(K^{(p)}, K^{(p-1)})$ is isomorphic to $\bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma$, we will actually want to work with a chain complex with chain groups

$$C_p^{CW}(K) := \bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma .$$

DEFINITION

Define two homomorphisms:

$$C_n^{CW}(K) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} H_n(K^{(n)}, K^{(n-1)})$$

Recall that $(B_\sigma^n, \partial B_\sigma^n) \xrightarrow{f_\sigma} (K^{(n)}, K^{(n-1)})$.

Denote by $[B_\sigma^n]$ the generator of $H_n(B_\sigma^n, \partial B_\sigma^n)$.

$$\Psi\left(\sum_\sigma n_\sigma \cdot \sigma\right) = \sum_\sigma n_\sigma (f_\sigma)_*([B_\sigma^n])$$

To define Φ, let $\phi_n : H_n(S^n, *) \to \mathbb{Z}$

be the unique homomorphism s.t.

$\phi_n([S^n]) = 1$.

Recall

bouquet of spheres

$$\begin{tikzcd}
(K^{(n)}, K^{(n-1)}) \arrow[r] \arrow[dr, "p_\sigma"'] & (K^{(n)}/K^{(n-1)}, *) \arrow[d] \\
& (S^n_\sigma, *)
\end{tikzcd}$$

Define

$$\Phi(\alpha) = \sum_{\sigma \in I_n} \phi_n\big((p_\sigma)_*(\alpha)\big)\sigma .$$

We have already proved that Ψ is an isomorphism (the iso from the last lecture) - check the details yourself.

CLAIM

$\Phi = \Psi^{-1}$.

Proof

It is enough to prove that $\Phi \circ \Psi = id$ since Ψ is an isomorphism.

Since $C_n^{cw}(X)$ is generated by the n-cells σ, it is enough to check $\Phi \circ \Psi(\sigma) = \sigma \quad \forall \sigma$.

$$\Phi\Psi(\sigma) = \Phi\left((f_\sigma)_*[B_\sigma^n]\right) =$$

$$= \sum_\tau \phi_n\left((p_\tau)_*(f_\sigma)_*[B_\sigma^n]\right)\tau \overset{*}{=}$$

Note that $p_\tau \circ f_\sigma = \begin{cases} \text{const. at } * & \tau \neq \sigma \\ id & \tau = \sigma \end{cases}$

So

$(*) = \phi_n([S^n]) \cdot \sigma = \sigma.$

$\blacksquare$

Define a boundary operator $C^{cw}_{n+1}(K) \xrightarrow{d_{n+1}} C^{cw}_n(K)$ by:

$$
\begin{tikzcd}
C^{cw}_{n+1}(K) \arrow[rr, "d_{n+1}"] \arrow[d, "\Psi_{n+1}"', "\cong"] & & C^{cw}_n(K) \\
H_{n+1}(K^{(n+1)}, K^{(n)}) \arrow[rr, "\beta_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & & H_n(K^{(n)}, K^{(n-1)}) \arrow[u, "\Phi_n"', "\cong"] \\
& \tilde{H}_n(K^{(n)}) \arrow[ur, "j_n"'] &
\end{tikzcd}
$$

Clearly, $d_n \circ d_{n+1} = 0$ because $\beta_n \circ \beta_{n+1} = 0$.

Let us write this differential explicitly. For $\sigma \in C^{cw}_{n+1}(K)$, write

$$d_{n+1}(\sigma) = \sum_{\tau \in I_n} \underbrace{[\tau : \sigma]}_{\in \mathbb{Z}} \cdot \tau$$

INCIDENCE NUMBER

$$d_{n+1}(\sigma) = \Phi\, \beta_{n+1} \Psi(\sigma) =$$

$$= \Phi\, j_n \circ \partial_{n+1} (f_\sigma)_* ([B^{n+1}_\sigma]) =$$

$$= \Phi\Big(j_n\, (f_{\partial\sigma})_* ([\partial B^{n+1}_\sigma])\Big)$$

takes you to hom of the pair

$$= \sum_{\tau \in I_n} \phi_n\Big((p_\tau)_* \big(j_n ((f_{\partial\sigma})_* [\partial B^{n+1}_\sigma])\big)\Big)\tau$$

collapses $(n-1)$-skeleton

$$= \sum_{\tau} \phi_n\Big(\big(\overbrace{p_\tau \circ f_{\partial\sigma}}^{(p_\tau)_* \circ j_n \circ (f_{\partial\sigma})_*}\big)_* [\partial B^{n+1}_\sigma]\Big)\tau$$

here we just use

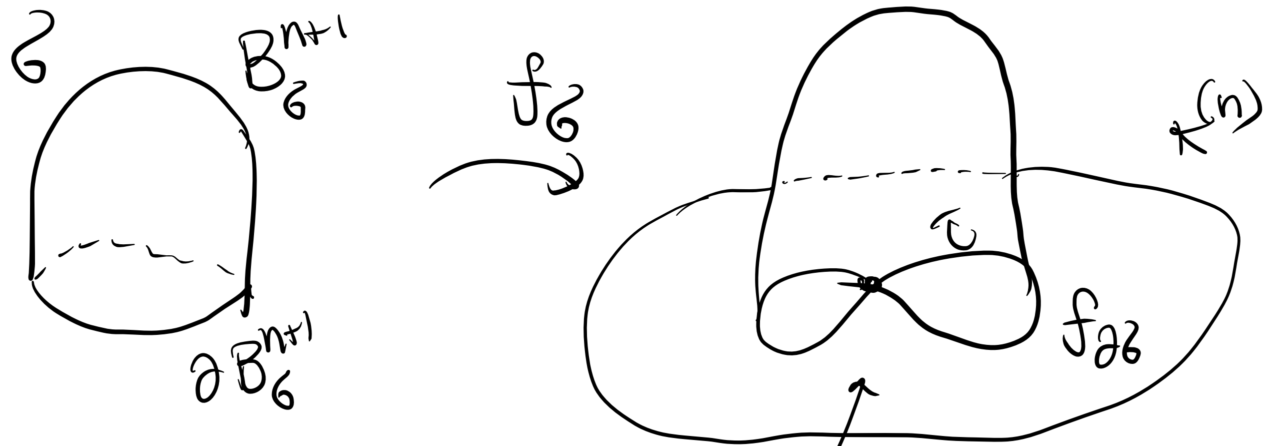
ϕ_n tells what multiple of the generator for τ we take

simplified notation

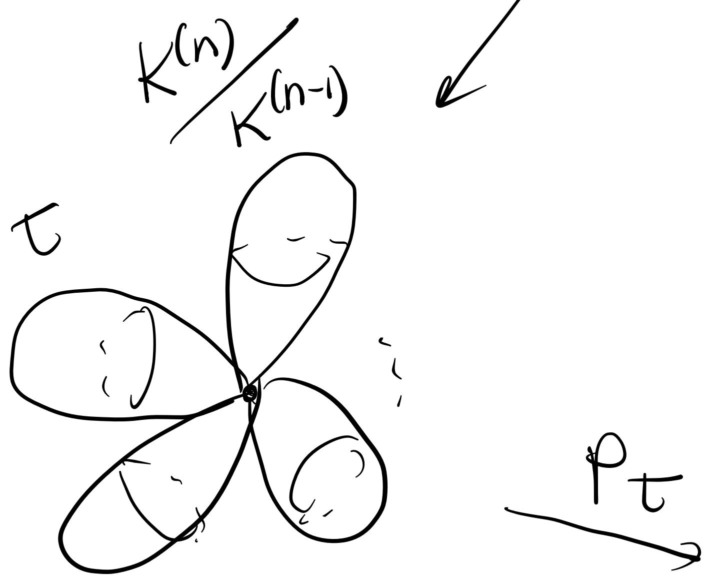
$$= \sum_{\tau} \deg(p_\tau \circ f_{\partial\sigma}) \cdot \tau$$

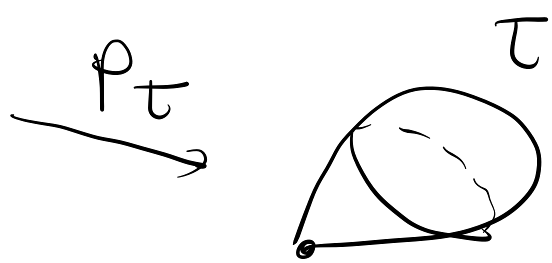
So, $[\tau : \sigma] = \deg(p_\tau \circ f_{\partial\sigma})$

What is happening geometrically?



the boundary is attached to many cells in $K^{(n)}$

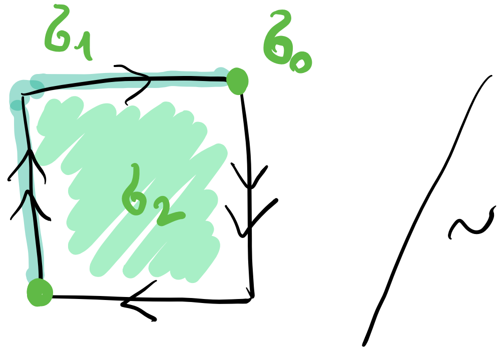




When $n=1$, $d_1 : C_1^{cw}(K) \to C_0^{cw}(K)$ is the same as the simplicial boundary map $\Delta_1(K) \to \Delta_0(K)$.

EXAMPLE

Take $K = \mathbb{R}P^2$.



This is a CW-complex:

one 0-cell σ_0

one 1-cell σ_1

one 2-cell σ_2

CW-chain complex

$$\cdots \to 0 \to \mathbb{Z}\sigma_2 \to \mathbb{Z}\sigma_1 \to \mathbb{Z}\sigma_0 \to 0$$

$d\sigma_0 = 0$

$d\sigma_1 = 0$ starting & ending point are the same

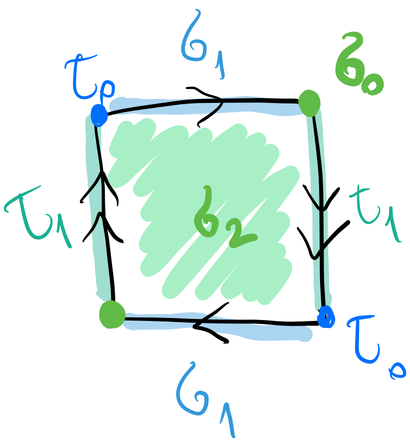
$d\sigma_2 = 2\cdot\sigma_1$ (or $-2\sigma_1$) sign depends on the choice of generators

$H_0(\mathbb{R}P^2) \cong \mathbb{Z}$

$H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$

$H_2(\mathbb{R}P^2) \cong 0$

Option #2 CW structure



0-cells

τ_0, σ_0

1-cells

τ_1, σ_1

2-cells

σ_2

Cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^2 \longrightarrow 0$$

$$d_2(\sigma_2) = 2\tau_1 + 2\sigma_1 = 2(\tau_1 + \sigma_1)$$

$$d_1(\tau_1) = \tau_0 - \sigma_0 \qquad \ker d_1 = \langle \tau_1 + \sigma_1 \rangle$$

$$d_1(\sigma_1) = \sigma_0 - \tau_0$$

$$H_0(\mathbb{R}P^2) = \langle \tau_0, \sigma_0 \rangle \big/ \operatorname{Im} d_1 \cong \langle \tau_0 - \sigma_0, \sigma_0 \rangle \big/ \langle \tau_0 - \sigma_0 \rangle \cong \mathbb{Z}$$

$$H_1(\mathbb{R}P^2) = \langle \tau_1 + \sigma_1 \rangle \big/ 2\langle \tau_1 + \sigma_1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_2(\mathbb{R}P^2) = \ker d_2 = 0$$

When doing exercises you don't have to worry about finding minimal cw-structure.

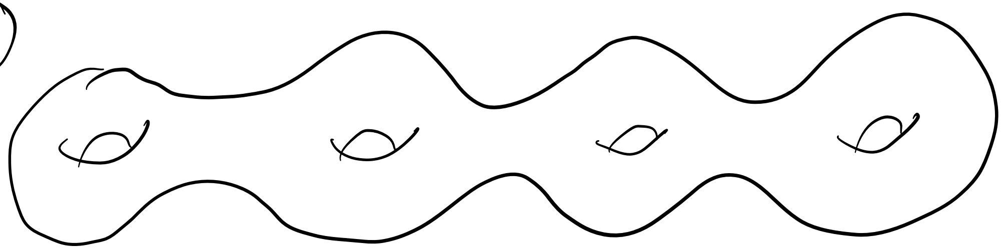
EXERCISE

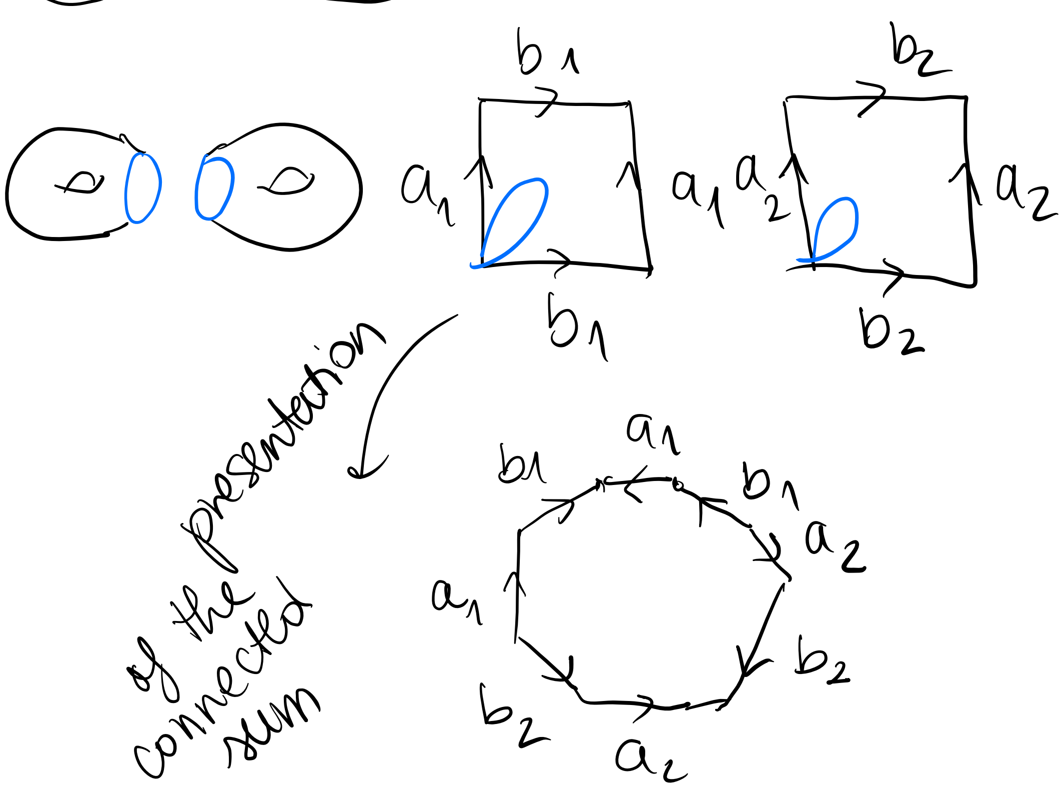
Let M_g be a closed orientable surface of genus g (connected sum of g many tori).

① Find CW-structure on M_g.

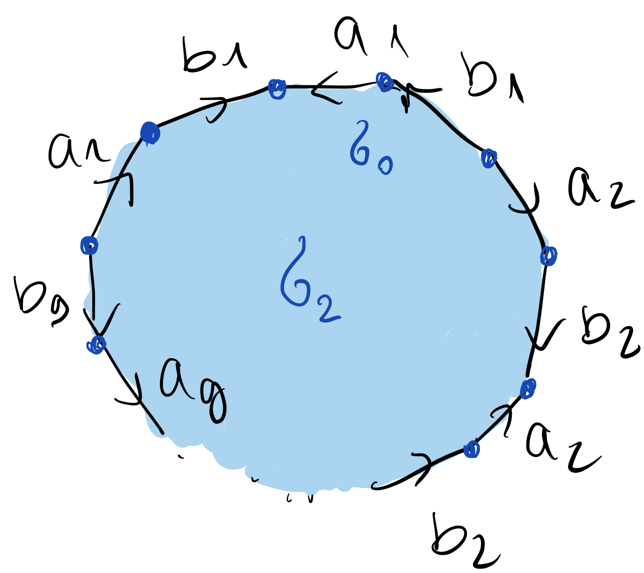
② Compute homology groups of M_g.

①





For M_g we have the following representation



CW-structure

1 0-cell σ_0

$2g$ 1-cells

$a_1, a_2, b_1, b_2, \dots a_g, b_g$

1 2-cell σ_2

② Cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

$$d_2(\sigma_2) = a_1 + b_1 - a_1 - b_1 + a_2 + b_2 - a_2 - b_2 + \dots$$
$$\dots + a_g + b_g - a_g - b_g = 0$$

$$d_1(a_i) = 0, \ d_1(b_i) = 0$$

So,

$$H_0(M_g) \cong \mathbb{Z}, \ H_1(M_g) \cong \mathbb{Z}^{2g}, \ H_2(M_g) \cong \mathbb{Z}.$$

WINTER 2016 Exam X, Y CW-complexes

ⓐ $k \in \mathbb{N}$, ω $(k-1)$-cell σ $(k+1)$-cell

Show that

$$\sum_{\tau \text{ k-cells}} [\omega : \tau][\tau : \sigma] = 0$$

Proof

$$d_{n+1}(\sigma) = \sum_{\tau} [\tau : \sigma] \tau$$

$$0 = d_n \circ d_{n+1}(\sigma) = \sum_{\tau} [\tau : \sigma] \, d_n(\tau)$$

$$= \sum_{\substack{\tau \\ \text{k-cells}}} \sum_{\substack{\omega \\ (k-1)\text{-cells}}} [\tau : \sigma][\omega : \tau] \, \omega =$$

$$= \sum_{\substack{\omega \\ (k-1)\text{-cells}}} \left(\sum_{\substack{\tau \\ \text{k-cells}}} [\tau : \sigma][\omega : \tau] \right) \omega$$

$$\Rightarrow \sum_{\substack{\tau \\ \text{k-cells}}} [\tau : \sigma][\omega : \tau] = 0$$