Proposition

$\sim$ is an equivalence relation on the set of all maps $X \to Y$.

Proof

- REFLEXIVE

$$H(x,t) = f(x) \quad \forall x \in X, \; t \in [0,1]$$

H is a homotopy between f & f $\Rightarrow f \sim f$.

- SYMMETRIC

$f \sim g$

$\exists H : X \times I \to Y \Rightarrow$

$H(x,0) = f(x)$

$H(x,1) = g(x)$

$F(x,t) = H^{-1}(x,t)$ is a homotopy from g to f, so $g \sim f$

- TRANSITIVE

$f \underset{F}{\sim} g$ & $g \underset{H}{\sim} h$.

$F * H$ is the homotopy from f to h. $\Rightarrow f \sim h$

NOTATION

X, Y spaces. Let $[X, Y]$ = set of homotopy classes of maps $X \to Y$

For pairs we write $[(X, A), (Y, B)]$.

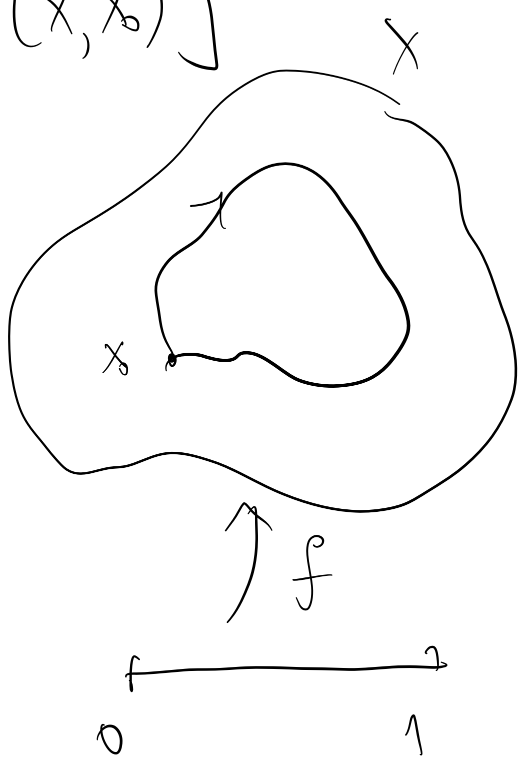
FUNDAMENTAL GROUP

$$\pi_1(X, x_0) = [(I, \partial I), (X, x_0)]$$

$\parallel$

homotopy classes of maps

$f : (I, \partial I) \to (X, x_0)$.



Since $I/\partial I \approx S^1$,

we have that

$$\pi_1(X, x_0) = [(S^1, *), (X, x_0)].$$

$\uparrow$ base point on S^1

Review
Exercise: Prove that $\pi_1(X, x_0)$ is a group.

HIGHER HOMOTOPY GROUPS

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$$

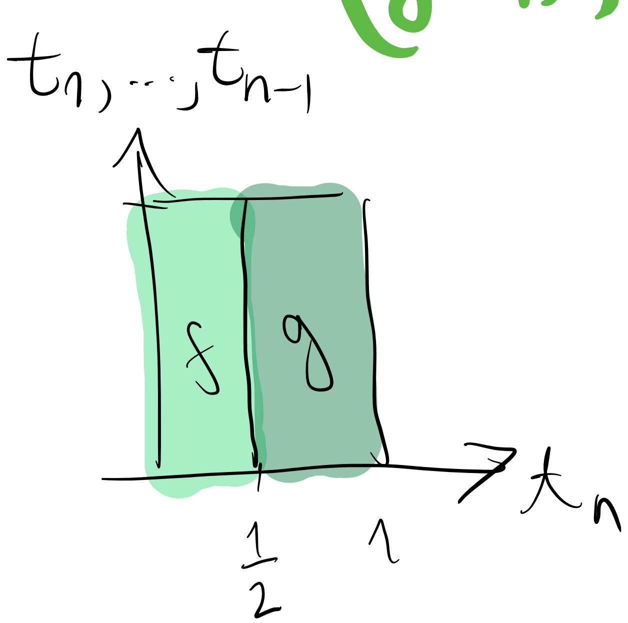
Since $I^n / \partial I^n \approx S^n$, $\pi_n(X, x_0) = [(S^n, *), (X, x_0)]$.

OPERATION

$$f, g : I^n \to X \qquad n \geq 2$$

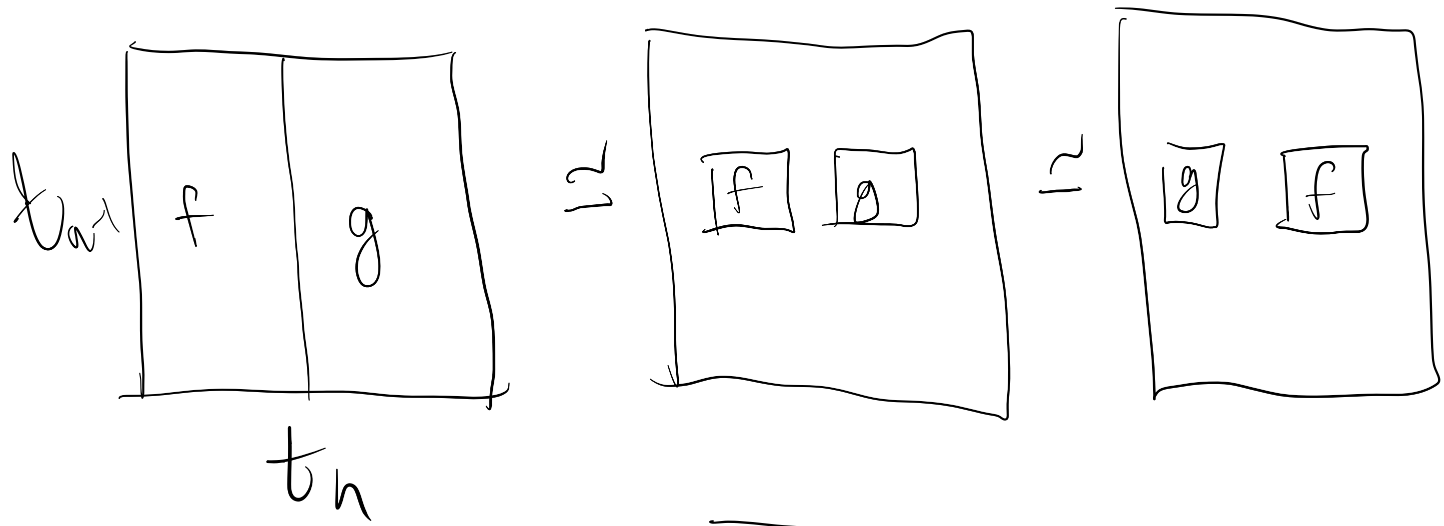
$$f + g\,(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n) & 0 \leq t_n \leq \frac{1}{2} \\ g(t_1, \ldots, t_{n-1}, 2t_n - 1) & \frac{1}{2} \leq t_n \leq 1 \end{cases}$$

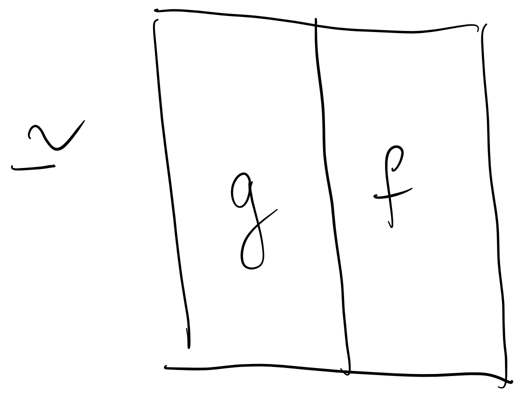
<u>Visually:</u>



So we define $[f]+[g]:=[f+g]$.

We use the additive operation since is commutative for $n\geq 2$, the homotopy $f+g \simeq g+f$ can be visualized as follows:





we use only the t_{n-1} & t_n coordinates:

the idea is to shrink the domains of f & g into smaller subcubes of I^n with the region outside mapping to the basepoint. Once this is done, we slide them past each other (so they remain disjoint) and interchange their positions. To finish the homotopy, we enlarge them back to their original size.

Further properties of $+$:

- ASSOCIATIVE
- IDENTITY ELEMENT

  $\text{CONST}: I^n \to \{x_0\}$
- INVERSE

  $-f(t_1, .., t_n) = f(t_1, .., 1-t_n)$

Equipped with this operation $\pi_n(X, x_0)$ is an abelian group.

Examples

(1) $\pi_n(S^k) = 0 \quad \forall n < k$

(3) $\pi_3(S^2) \cong \mathbb{Z}$

(2) $\pi_n(S^n) \cong \mathbb{Z}$

$\pi_n(S^k)$ for general $n > k$ is unknown.

<u>Drawback</u>: homotopy groups are very hard to compute in general.

<u>Alternative</u>: HOMOLOGY GROUPS

The most important homology theory in algebraic topology and the one we will be studying almost exclusively, is called SINGULAR HOMOLOGY. But since the technical apparatus of singular homology is somewhat complicated, we will start with simplicial homology.

We will define it for Δ-complexes (Hatcher), which are a slight generalization of simplicial complexes.

DELTA COMPLEXES

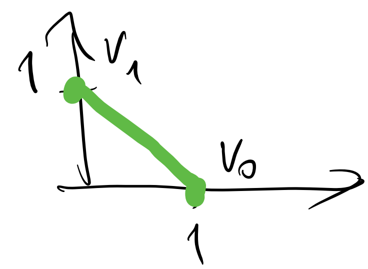
Definition

The **STANDARD n-dimensional SIMPLEX** (or n-simplex) is the topological space

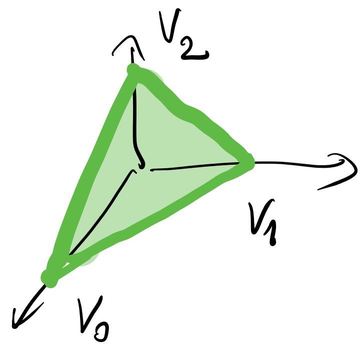
$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \ \forall i\}.$$

Example. Δ^0 is a point • v_0

Δ^1 is a line segment



Δ^2 is a triangle



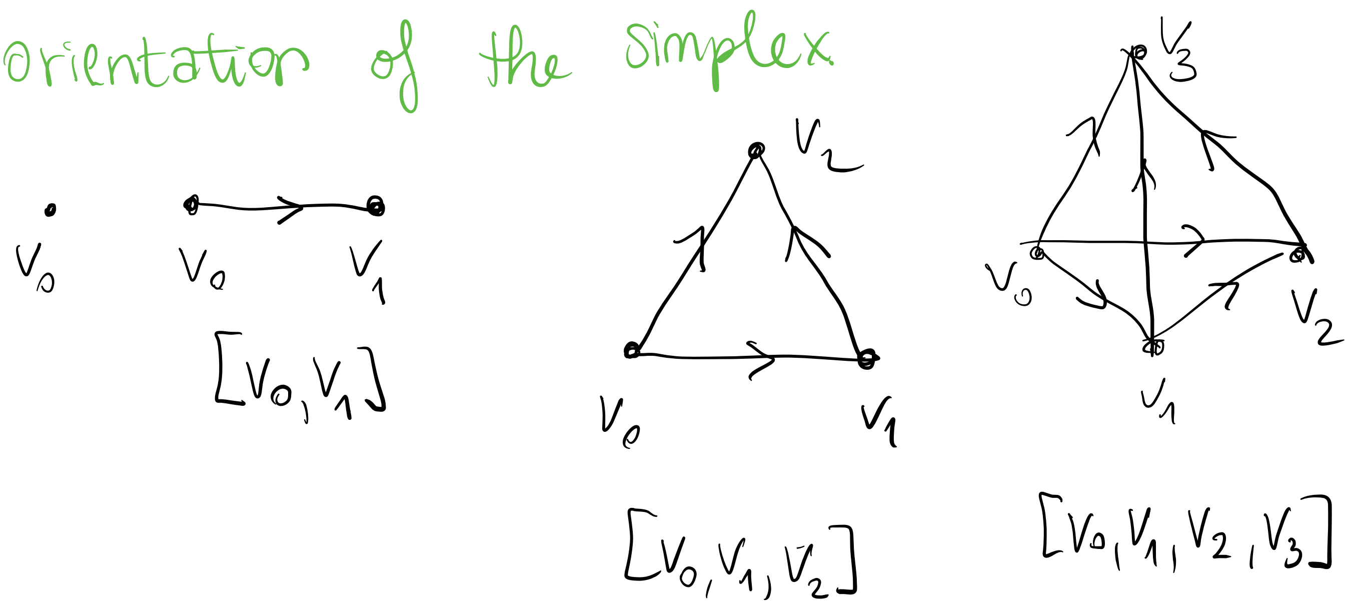
An n-simplex is the smallest convex set in a Euclidean space $\mathbb{R}^m$ containing $n+1$ points $V_0, V_1, \dots, V_n$ that do not lie in a hyperplane of dimension less than n (or equivalently, such points that the difference vectors $V_1 - V_0, \dots, V_n - V_0$ are linearly independent).

The points V_i are called **VERTICES** of the simplex and the simplex itself is denoted by $[v_0, \dots, v_n]$.

Example

the vertices of the standard n-simplex are the unit vectors along the coordinate axes.

ORDERING OF THE VERTICES IS IMPORTANT as it determines the orientation of the simplex.



The ordering also determines a canonical linear homomorphism from the standard n-simplex Δ^n onto any other simplex $[v_0, \dots, v_n]$, preserving the order of vertices, namely:

$$(t_0, \dots, t_n) \longmapsto \sum_{i=0}^{n} t_i v_i$$

t_i are called barycentric coordinates of $\sum t_i v_i$ in $[v_0, \dots, v_n]$

Definition

If we delete one of the $n+1$ vertices of an n-simplex $[v_0, \dots, v_n]$, then the remaining n vertices span an $(n-1)$ simplex, called a **FACE** of $[v_0, \dots, v_n]$. It is ordered according to the order in $[v_0, \dots, v_n]$.