Definition

The union of all faces of Δ^n is the **BOUNDARY** of Δ^n, written $\partial\Delta^n$.

The **OPEN SIMPLEX** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.

Gluing simplices together in a principled way yields a **Δ-COMPLEX**.

Definition

A **Δ-COMPLEX STRUCTURE on a space X** is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open $\Longleftrightarrow$

$\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

rules out trivialities like regarding all the points as individual vertices

One consequence of (iii) is that X can be built as a quotient space of a collection of disjoint simplices inductively; start with a discrete set of vertices, attach edges to produce graphs, attach triangles or 2-simplices, etc.

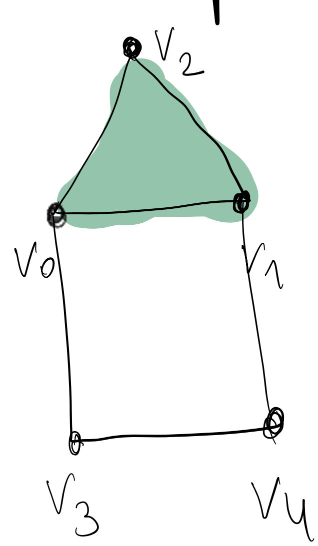
SIMPLICIAL HOMOLOGY

Definition

Let $\Delta_n(X)$ be the **free abelian group with basis the open n-simplices e^n_α of X.**

Elements of $\Delta_n(X)$ are called **n-chains** and can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in \mathbb{Z}$. Equivalently, we could write $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e_α^n, with image the closure of e_α^n.

Example



$v_1 + v_2 + v_3$, v_0, $v_0 + v_4$ are all examples of 0-chains.
$[v_0, v_1] + [v_1, v_4] - [v_3, v_4] - [v_0, v_3]$ is a 1-chain, as is $[v_0, v_1] + [v_1, v_4] + [v_3, v_4] + [v_0, v_3]$.

the boundary of $[v_0, v_1, v_2]$
consists of 1-simplices $[v_0, v_1], [v_1, v_2], [v_0, v_2]$
We might say that the boundary is the 1-chain formed by the sum of the faces $[v_0, v_1], [v_1, v_2], [v_0, v_2]$. However, we must take orientations into account.

Definition

The **BOUNDARY HOMOMORPHISM**

$$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$$

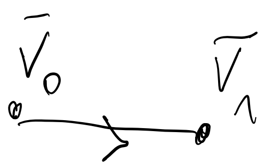
is specified on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

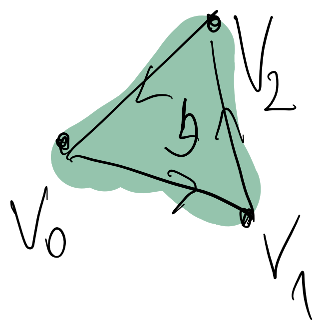
this is well-defined since each restriction $\sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

is the characteristic map of an $(n-1)$-simplex

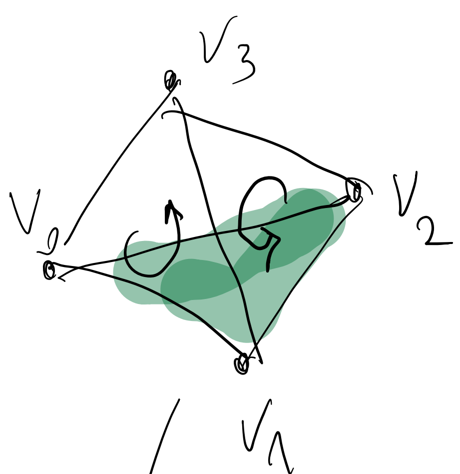
Examples



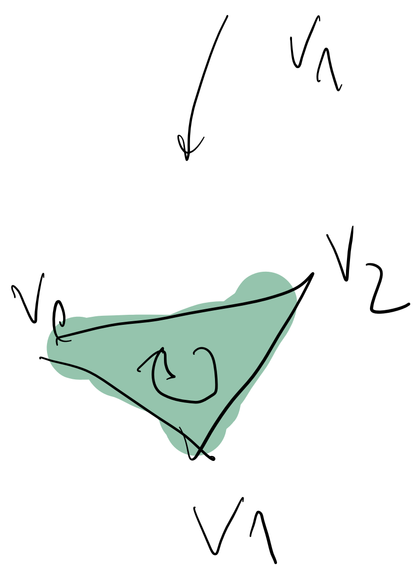
$$\partial_1[v_0, v_1] = [v_1] - [v_0]$$



$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\partial_3[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$



$$[v_0, v_2, v_1] = -[v_0, v_1, v_2]$$

Lemma

The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.

Proof

$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$, and hence

$$\partial_{n-1}\partial_n(\sigma) = \partial_{n-1}\Big(\sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\Big)$$

$$= \sum_i (-1)^i \partial_{n-1} \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} =$$

$$= \sum_i (-1)^i \Big(\sum_{j<i} (-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}$$

$$+ \sum_{j>i} (-1)^{j-1} \sigma|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}\Big)$$

$$= \sum_i \sum_{j<i} (-1)^{i+j} \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}$$

$$+ \sum_i \sum_{j>i} (-1)^{i+j-1} \sigma|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}$$

The latter two summations cancel

since after switching i & j in the second sum, it becomes the negative of the first.

$\blacksquare$

The algebraic situation we have now is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

with $\partial_n \circ \partial_{n+1} = 0$ for each n.

Such a sequence is called a **CHAIN COMPLEX**. NOTATION $(C_\bullet, \partial_\bullet)$

The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$, where