since after switching $i$ & $j$ in the second sum, it becomes the negative of the first.

$\blacksquare$

The algebraic situation we have now is a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \to \cdots \to C_1 \xrightarrow{\partial_n} C_0 \xrightarrow{\partial_0 = 0} 0$$

with $\partial_n \circ \partial_{n+1} = 0$ for each $n$.

Such a sequence is called a **CHAIN COMPLEX**. NOTATION $(C_\bullet, \partial_\bullet)$

The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$, where

$\operatorname{Im} \partial_{n+1}$ denotes the image of $\partial_{n+1}$
& $\operatorname{Ker} \partial_n$ the kernel of $\partial_n$.

Definition
Let $(C_\bullet, \partial_\bullet)$ be a chain complex.
The **n-th homology group of $(C_\cdot, \partial_\cdot)$**

$$H_n = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$$

Elements of $\operatorname{Ker} \partial_n$ are called **cycles** and elements of $\operatorname{Im} \partial_{n+1}$ **boundaries**. Elements of $H_n$ are cosets of $\operatorname{Im} \partial_{n+1}$, called **homology classes**. Two cycles representing the same homology class are said to be **homologous**.

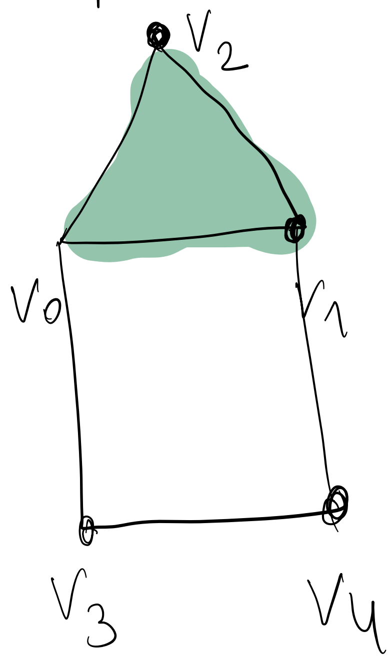
## Definition

When $C_n = \Delta_n(X)$, the homology group $\ker\partial_n / \operatorname{Im}\partial_{n+1}$ is denoted by $H_n^{\Delta}(X)$ and called the **$n$th simplicial homology group of $X$.**

Intuition: $H_n^{\Delta}(X)$ captures the information about $n$-dim holes in $X$.

Example 1 Homology groups of $X$

$X$




$\Delta_0(X) = \langle v_0, v_1, v_2, v_3, v_4 \rangle$

$\Delta_1(X) = \langle [v_0, v_2], [v_0, v_1], [v_0, v_3], [v_1, v_2], [v_1, v_4], [v_3, v_4] \rangle$

$\Delta_2(X) = \langle [v_0, v_1, v_2] \rangle$

$[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ $\quad \Delta_3(X) = 0$

Chain complex associated to X

$$\to 0 \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{0} 0$$

$$\partial_2([v_0, v_1, v_2]) =$$
$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$\ker \partial_2$ is trivial, so $H_2(X) = 0$.

Also $H_i(X) = 0$ for $i \geq 3$.

Kernel of $\partial_1$:

$$\partial_1 [v_0, v_2] = v_2 - v_0$$
$$\partial_1 [v_0, v_1] = v_1 - v_0$$
$$\partial_1 [v_0, v_3] = v_3 - v_0$$
$$\partial_1 [v_1, v_2] = v_2 - v_1$$
$$\partial_1 [v_1, v_4] = v_4 - v_1$$
$$\partial_1 [v_3, v_4] = v_4 - v_3$$

$$a(v_2-v_0)+b(v_1-v_0)+c(v_3-v_0)+d(v_2-v_1)+$$

$$e\,(v_4-v_1)+f\,(v_4-v_3)=0$$

$$v_0\,(\overset{(1)}{-a-b-c})+v_1\,(\overset{(2)}{b-d-e})$$

$$+v_2\,(\overset{(3)}{a+d})+v_3\,(\overset{(4)}{c-f})+v_4\,(\overset{(5)}{e+f})=0$$

(4) $f=c$

(3) $d=-a$

(5) $e=-f=-c$

(1) $b-d-e=0$

$$b=d+e=$$

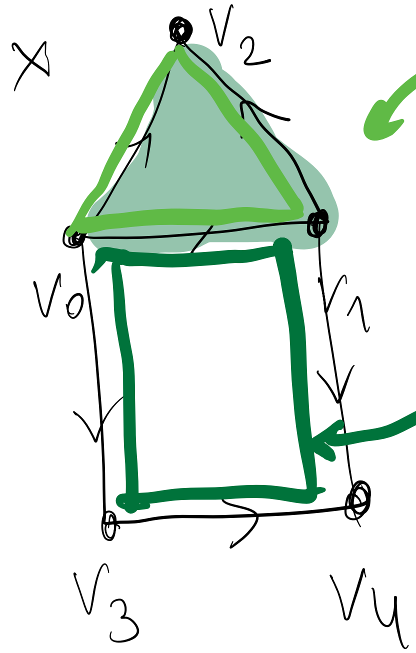
$$=-a-c$$

(2) $(-a-c)-(-a)-(-c)=0$

$$(a,-a-c,c,-a,-c,c)=$$

$$=a(1,-1,0,-1,0,0)+c(0,-1,1,0,-1,1)$$

$$\ker \partial_1 = \langle [v_0,v_2] - [v_0,v_1] - [v_1,v_2],$$
$$-[v_0,v_1] + [v_0,v_3] - [v_1,v_4] + [v_3,v_4] \rangle$$




$$[v_0,v_2] - [v_0,v_1] - [v_1,v_2]$$

$$-[v_0,v_1] + [v_0,v_3] - [v_1,v_4] + [v_3,v_4]$$

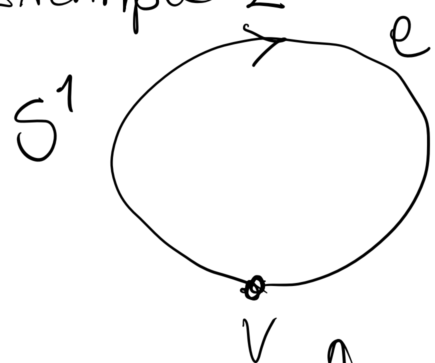
$$\operatorname{Im} \partial_2 = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$$

$$H_1^{\Delta}(X) = \ker \partial_1 / \operatorname{Im} \partial_2 \cong \langle [v_0,v_1] + [v_0,v_3] - [v_1,v_4] + [v_3,v_4] \rangle$$
$$\cong \mathbb{Z}$$

the space X has one hole

This is an example of a special type of a $\Delta$-complex, called a **SIMPLICIAL COMPLEX**. Simplicial complexes can be encoded combinatorially and software exists to compute their homology groups (coefficients are taken from a finite field, so that the computation reduces to linear algebra).

Example 2




$\Delta$-complex structure on $S^1$

$\Delta_0(S^1) = \langle v \rangle$

$\Delta_1(e) = \langle e \rangle$

$\Delta_i(x) = 0 \quad \forall\, i \geq 2$

$$\Delta_1(e) \xrightarrow{\partial_1} \Delta_0(S^1) \to 0 \qquad \partial_1(e) = v - v = 0$$

chain complex associated to $S^1$

$$H_1^{\Delta}(X) = \ker \partial_1 = \langle e \rangle \cong \mathbb{Z}$$

$$H_0^{\Delta}(X) = \Delta_0(S^1) / \operatorname{Im} \partial_1 \cong \langle v \rangle \cong \mathbb{Z}$$

$$H_i^{\Delta}(X) = 0 \text{ for } i \geq 2$$

Questions: Are the groups $H_n^{\Delta}(X)$ independent of the choice of $\Delta$-complex structure on $X$?

If two $\Delta$-complexes are homeomorphic, do they have isomorphic homology groups?

To answer these, we first develop a more general theory of singular homology groups, which have the added benefit of being defined for all spaces, not just $\Delta$-cx.