SINGULAR HOMOLOGY

Definition

A **SINGULAR n-simplex** in X is a map $\sigma : \Delta^n \to X$.

The word singular is used to express the idea that σ need not be a nice embedding, but can have 'singularities' where its image does not look like a simplex. All that is required is that σ is continuous.

Definitions

$S_n(X)$ is the free abelian group generated by all the singular n-simplices $\sigma : \Delta^n \to X$ of X.

We call $S_n(X)$ the **GROUP OF SINGULAR n-CHAINS** of X.

A **singular n-chain** is a (finite) formal sum

$$\sum_{\sigma:\Delta^n\to X} n_\sigma\cdot\sigma,\quad n_\sigma\in\mathbb{Z}.$$

The **BOUNDARY MAP** is defined by the same formula as before:

$$\partial_n(\sigma)=\sum_i(-1)^i\,\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$ is regarded as a map $\Delta^{n-1}\to X$ via the canonical identification of $[v_0,\dots,\hat{v}_i,\dots,v_n]$ with Δ^{n-1}, preserving the ordering of the vertices

The proof that $\partial_n \circ \partial_{n+1} = 0$ works the same as in the simplicial homology case.

Notation: We often denote all ∂_n by ∂ and write $S_n(x) \xrightarrow{\partial} S_{n-1}(x)$ & $\partial \circ \partial = 0$.

Definition

$(S_\bullet(x), \partial_\bullet)$ is a chain complex.

The SINGULAR HOMOLOGY groups are

$$H_n(x) = \operatorname{Ker} \partial_n \big/ \operatorname{Im} \partial_{n+1}$$

cycles

$\operatorname{Ker} \partial_n = Z_n(x)$

$\operatorname{Im} \partial_{n+1} = B_n(x)$

boundaries

Example

X point. What are the homology groups of X?

For each dimension $n \geq 0$ we have exactly <u>one</u> singular simplex

$\sigma_n : \Delta^n \to X$. So, $S_n(X) = \mathbb{Z} \cdot \sigma_n$.

We now calculate $\partial_n : S_n(X) \to S_{n-1}(X)$.

$\partial_n(\sigma_n) =$ an alternating sum of $(n+1)$ elements each of which is σ_{n-1}

$$\partial_n(\sigma_n) = \begin{cases} 0 & n = \text{odd} \\ \sigma_{n-1} & n \text{ is even} > 0 \\ 0 & n = 0 \end{cases}$$

$$\begin{tikzcd} \cdots \arrow[r] & S_3(X) \arrow[r, "\partial_3"] \arrow[d, phantom, "\cong"] & S_2(X) \arrow[r, "\partial_2"] \arrow[d, phantom, "\cong"] & S_1(X) \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong"] & S_0(X) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ \cdots \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

∂_n is an isomorphism for even $n > 0$ and the zero map when n is odd.

Cycles:

$$Z_n(X) = \begin{cases} \mathbb{Z} & n = \text{odd} \\ 0 & n = \text{even} \ \& >0 \\ \mathbb{Z} & n = 0 \end{cases}$$

Boundaries

$$B_n(X) = \begin{cases} \mathbb{Z} & n = \text{odd} \\ 0 & n = \text{even} \ \& n>0 \\ 0 & n = 0 \end{cases}$$

$$H_n(X) = \begin{cases} 0 & n = \text{odd} \\ 0 & n = \text{even} \ \& >0 \\ \mathbb{Z} & n = 0 \end{cases}$$

$$= \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

FUNCTORIAL PROPERTIES

Let $f: X \to Y$ be a map between the spaces X & Y. For every

singular n-simplex $\sigma : \Delta^n \to X$,
we get a new singular simplex
induced by $f \circ \sigma : \Delta^n \to Y$.
Extending linearly we get
a homomorphism defined by

$$f_n = S_n(f) : S_n(X) \to S_n(Y)$$

$$f_n\left(\sum_\sigma n_\sigma \cdot \sigma\right) = \sum_\sigma n_\sigma (f \circ \sigma).$$

Proposition

$$f_{n-1} \circ \partial_n^X = \partial_n^Y \circ f_n \quad (f_c \circ \partial = \partial \circ f_c).$$

Proof

$$f_{n-1} \circ \partial_n(\sigma) = f_{n-1}\left(\sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right)$$

$$= \sum_i (-1)^i f \circ \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$= \partial_n (f \circ \sigma) = \partial_n \circ f_n(\sigma) \quad \blacksquare$$

Thus we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & S_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & S_{n-1}(X) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & S_{n+1}(Y) \arrow[r, "\partial_{n+1}"'] & S_n(Y) \arrow[r, "\partial_n"'] & S_{n-1}(Y) &
\end{tikzcd}$$

Homomorphisms $f_c : S_\bullet(X) \to S_\bullet(Y)$ that satisfy $f_c \circ \partial = \partial \circ f_c$ are called **CHAIN MAPS** from the singular chain complex of X to that of Y.