Thus we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & S_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & S_{n-1}(X) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & S_{n+1}(Y) \arrow[r, "\partial_{n+1}"'] & S_n(Y) \arrow[r, "\partial_n"'] & S_{n-1}(Y) &
\end{tikzcd}$$

Homomorphisms $f_C : S_\bullet(X) \to S_\bullet(Y)$ that satisfy $f_C \circ \partial = \partial \circ f_C$ are called **CHAIN MAPS** from the singular chain complex of X to that of Y.

COROLLARY

① $f_c(Z_n(X)) \subset Z_n(Y)$

$f_c(B_n(X)) \subset B_n(Y)$

In particular f induces, via f_c, a homomorphism $f_*: H_n(X) \to H_n(Y)$, by $f_*([c]) = [f_c(c)]$.

② $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow$

$$(g \circ f)_* = g_* \circ f_* : H_n(X) \to H_n(Z)$$

$$\& \quad (id_X)_* = id_{H_n(X)}.$$

Proof

① If $c \in Z_n(X)$ (ie. $\partial c = 0$) $\Rightarrow$

$$\partial f_c(c) = f_c(\underbrace{\partial c}_{0}) = 0$$

$\Rightarrow f_c(c) \in Z_n(Y)$.

If $c = \partial d$, $d \in S_{n+1}(X) \Rightarrow$

$f_c(c) = f_c(\partial d) = \partial f_c(d) \in B_n(Y)$.

$\Rightarrow f$ induces a homomorphism

$$\underbrace{Z_n(X)/B_n(X)}_{H_n(X)} \xrightarrow{f_*} \underbrace{Z_n(Y)/B_n(Y)}_{H_n(Y)}$$

2) Exercise.

Notation:

f_*, $H(f): H_n(X) \to H_n(Y)$

f_* is called the map induced by

f in homology

COROLLARY

If $f: X \to Y$ is a homeomorphism, then $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Proof

Put $g := f^{-1}: Y \to X$, so $f \circ g = id_Y$, $g \circ f = id_X$.

$$id_{H_n(Y)} = (id_Y)_* = (f \circ g)_* =$$

$$= f_* \circ g_* : H_n(Y) \to H_n(Y)$$

$$id_{H_n(X)} = (id_X)_* = (g \circ f)_* =$$

$= g_* \circ f_* : H_n(X) \to H_n(X)$.

THE ZEROTH HOMOLOGY GROUP (Bredon)

X space. What is $H_0(X)$?

A 0-simplex $\sigma : \underset{\substack{\| \\ \text{point}}}{\Delta^0} \to X$ is just

a choice of a point in X.

A 0-chain in X is a finite formal

sum $c = \sum\limits_{x \in X} n_x \cdot x$. Clearly, $\partial(c) = 0$.

Define $\varepsilon(c) = \sum\limits_{x \in X} n_x \in \mathbb{Z}$.

Easy to check $\varepsilon : S_0(X) \to \mathbb{Z}$ is

a homomorphism.

Let σ be a singular 1-simplex,

$\sigma : \Delta^1 \to X$, Put $x_0 = \sigma(0), x_1 = \sigma(1)$.

$$\partial\sigma = \sigma(1) - \sigma(0) = x_1 - x_0$$

$$\Rightarrow \varepsilon(\partial\sigma) = 1 - 1 = 0.$$

So for each 1-dimensional chain d we have $\varepsilon(\partial d) = 0$. $\Rightarrow \varepsilon(B_0(X)) = 0$.

It follows that ε induces a homomorphism

$$\varepsilon_* : H_0(X) \to \mathbb{Z}.$$

Both ε and ε_* are called

AUGMENTATION.

Theorem

If $X \neq \emptyset$ is path-connected, then $\varepsilon_* : H_0(X) \to \mathbb{Z}$ is an isomorphism. Moreover, $H_0(X) \cong \mathbb{Z} \cdot [x]$ where $x \in X$ is any point.

Proof

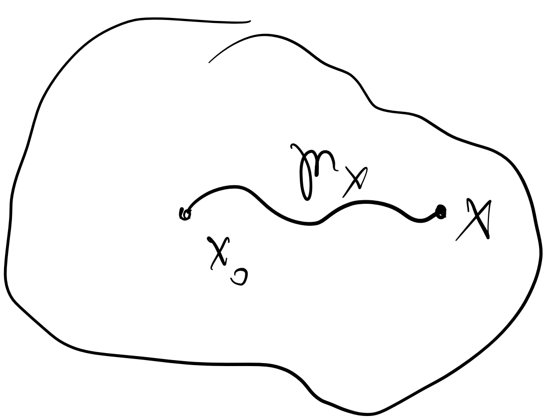
Clearly, $\varepsilon_*([y]) = 1 \quad \forall y \in X$.

So ε_* is surjective.

We must show that ε_* is injective.

Fix $x_0 \in X$. $\forall x \in X$ choose a path

$\mu_x : I \to X$ with $\mu_x(0) = x_0$, $\mu_x(1) = x$.



If we view m_x as a singular 1-simplex, $\partial m_x = x - x_0 \in S_0(X)$.

Let $c \in S_0(X)$, $c = \sum_x n_x \cdot x$, be a 0-chain. Assume $[c] \in \ker(\varepsilon_*)$, we $\sum_x n_x = 0$.

Consider the 1-chain $\sum n_x m_x \in S_1(X)$.

$$\partial\left(\sum n_x m_x\right) = \sum n_x (x - x_0) = \sum n_x \cdot x - \underbrace{\left(\sum n_x\right)}_{=0} \cdot x_0 = c \Rightarrow c \in B_0(X).$$

$$\Rightarrow [c] = 0 \Rightarrow \ker \varepsilon_* = 0.$$

Proposition

Let $X \neq \emptyset$ and denote by $\mathcal{C}$ the set of path-connected components of X. $\forall \alpha \in \mathcal{C}$, denote by $X_\alpha \subset X$ the path-connected component corresponding to α. Then

$$H_n(X) \cong \bigoplus_{\alpha \in \mathcal{C}} H_n(X_\alpha),$$ where

this isomorphism is induced by inclusions. In particular,

$$H_0(X) \cong \bigoplus_{\alpha \in \mathcal{C}} H_0(X_\alpha)$$

$$\cong \bigoplus_{\alpha \in \mathcal{C}} \mathbb{Z}$$

Proof

Since a singular simplex always has a path-connected image, $S_n(X)$ splits as the direct sum of its subgroups $S_n(X_\alpha)$. The boundary maps ∂_n preserve this direct sum decomposition, taking $S_n(X_\alpha)$ to $S_{n-1}(X_\alpha)$, so $\ker \partial_n$ and $\operatorname{Im} \partial_{n+1}$ split similarly as direct sums.

Hence, homology groups also split,

$$H_n(X) \cong \bigoplus_{\alpha \in \mathcal{C}} H_n(X_\alpha).$$

REDUCED HOMOLOGY GROUPS

(Hatcher)

It is often convenient to have a slightly modified version of homology for which a point has trivial homology in all dimensions, including 0. This is done by defining REDUCED HOMOLOGY GROUPS $\tilde{H}_n(X)$ to be the homology groups of the AUGMENTED CHAIN COMPLEX

$$\cdots \to S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the augmentation.
Usually, we require X to be nonempty to avoid a non-trivial homology group in dim -1.

Then $\tilde{H}_0(X)$ is the kernel of ε^* so that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

For $n > 0$ $H_n(X) \cong \tilde{H}_n(X)$.

THE FIRST HOMOLOGY GROUP

We now establish a link between the present subject of topology and our previous discussion of homotopy. In particular, what the connection between the fundamental group of a space & H_1 of a space is.

THEOREM

Let X be a path-connected space. Fix a base point $x_0 \in X$. Put

$$G := \pi_1(X, x_0)$$

Then

$$H_1(X) \cong G^{ab} := G/[G,G].$$

abelianization

Recall that $[G,G]$ is the subgroup generated by all the commutators, ie elements of the form $[g,h] = g^{-1}h^{-1}gh$.

Examples

(1) $$\pi_1(S^n, *) \cong \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$$