THE FIRST HOMOLOGY GROUP

We now establish a link between the present subject of homology and our previous discussion of homotopy. In particular, what the connection between the fundamental group of a space & H_1 of a space is.

THEOREM [HUREWICZ THEOREM]
Let X be a path-connected space. Fix a base point $x_0 \in X$. Put

$$G := \pi_1(X, x_0)$$

Then

$$H_1(X) \cong G^{ab} := G/[G,G].$$

abelianization

Recall that $[G,G]$ is the subgroup generated by all the commutators, ie elements of the form $[g,h] = g^{-1}h^{-1}gh$.

Examples

1. $\pi_1(S^n, *) \cong \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$

$\Rightarrow H_1(S^n) = \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$

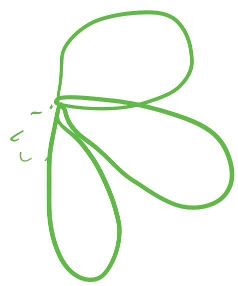
2. Recall that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

then $\pi_1(T) = \pi_1(S^1 \times S^1)$

$$\cong \pi_1(S^1) \times \pi_1(S^1)$$

$$\cong \mathbb{Z} \times \mathbb{Z}$$

$\Rightarrow H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

③ $H_1\Big($  $\Big) \cong \big($ free non-abelian group on n letters $\big)^{ab}$

π_1

bouquet of n circles

$$\cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n\text{-times}}$$

Proving the Hurewicz theorem

Some notation

- homotopy classes $\{\dots\}$, homology classes [...]

- $f \underset{\pi}{\simeq} g$ means that f & g are homotopic & $f \underset{H}{\simeq} g$ means homologous

Lemma 1

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$. Consider the 1-chain

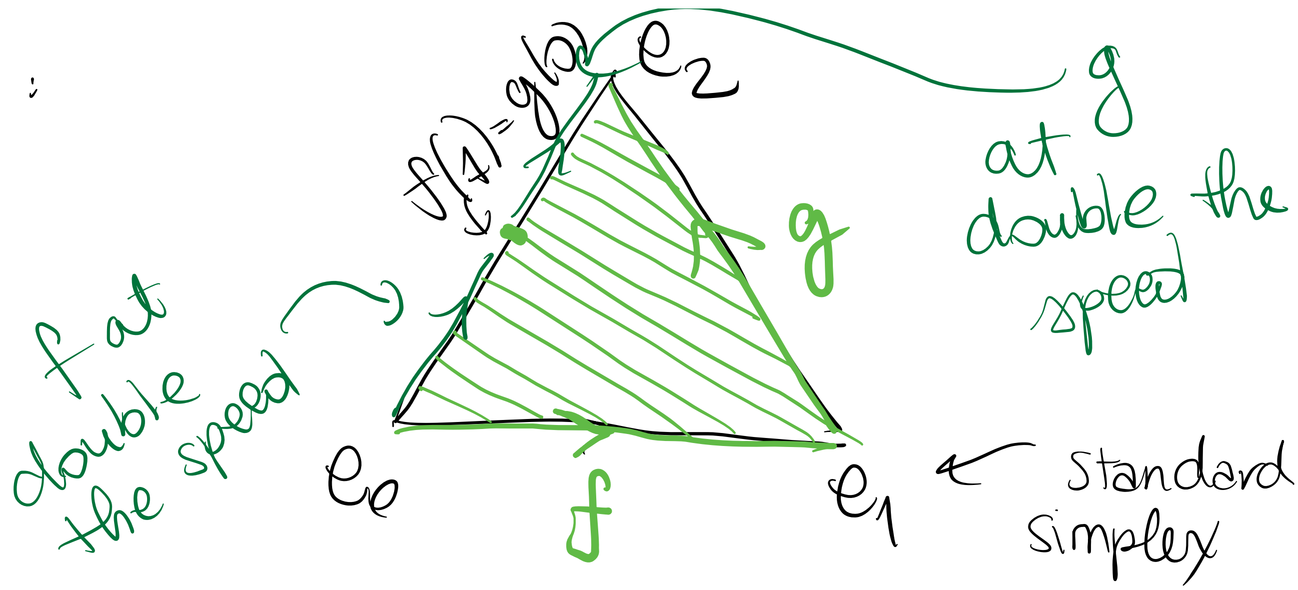
$$c := f * g - f - g \in S_1(X).$$

Then c is a boundary (hence $[c] = 0 \in H_1(X)$).

Proof of Lemma 1

Define $\sigma : \Delta^2 \longrightarrow X$ as follows:

Δ^2 :



- On the edge e_0, e_1 let it be f
- on the edge e_1, e_2 let it be g

Extend σ to the rest of Δ^2 s.t. on each segment in Δ^2, which is perpendicular to $e_0 e_2$, σ is constant.

So on $e_0 e_2$ we get that σ is $f * g$.

Let us calculate $\partial\sigma$:

$$\partial\sigma = (-1)^0 \sigma\big|_{[e_1,e_2]} + (-1)^1 \sigma\big|_{[e_0,e_2]} + (-1)^2 \sigma\big|_{[e_0,e_1]}$$

$$= g - f*g + f = f + g - f*g$$

$\Rightarrow$ $f*g - f - g$ is a boundary.

Lemma 2

① The constant path $c : I \to X$ is a boundary.

② Let $f : I \to X$ be a path. Then the 1-chain $f + f^{-1}$ is a boundary.

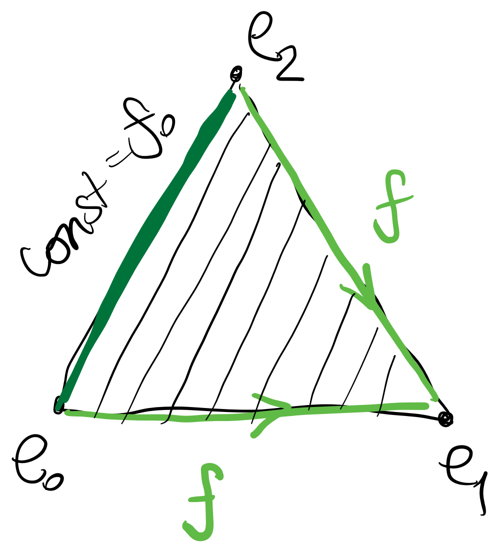
Proof of Lemma 2

① Define $\tau : \Delta^2 \to X$ to be the constant simplex (constant at the same point as c).

$\partial \tau = c - c + c = c$

(2) Define $\sigma : \Delta^2 \to X$ by defining σ to be on the edge e_0e_1 as well as e_2e_1.

Extend σ to the rest of Δ^2 by setting it to be constant on each segment parallel to e_0e_2.



$$\partial\sigma = f^{-1} - \text{const} + f$$

Since the constant edge is also a boundary by (1) $\partial\tau$,

$$\partial(\sigma + \tau) = f^{-1} + f$$

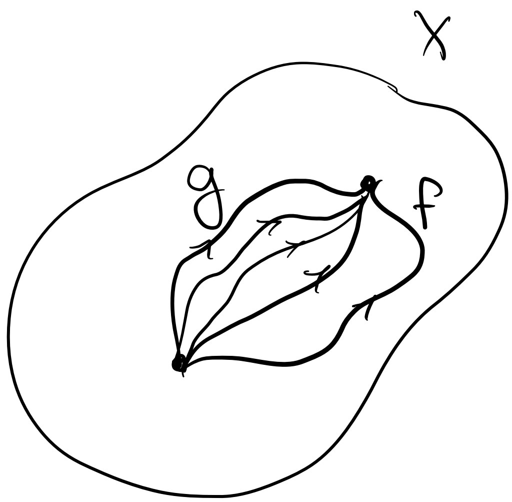
$f^{-1} + f$ is also a boundary.

Lemma 3

If $f, g : I \to X$ are paths with $f(0) = g(0)$, $f(1) = g(1)$ and $f \underset{\pi}{\simeq} g$ rel ∂I, then $f \underset{H}{\simeq} g$.

Proof

Let $F : I \times I \to X$ be a homotopy rel ∂I between f and g. We have

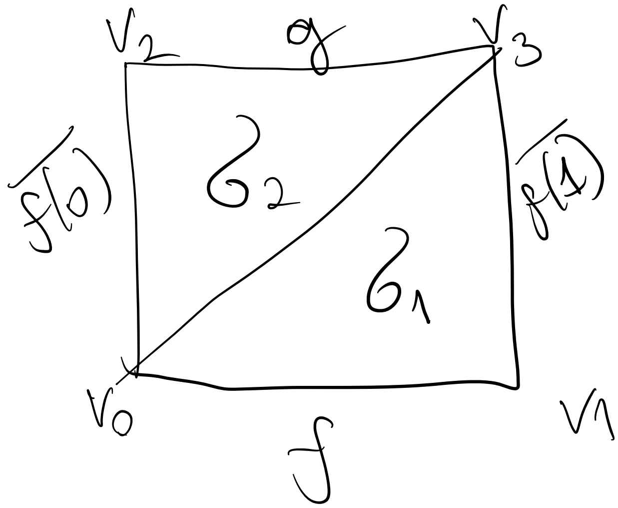


$$F\big|_{0 \times I} = \text{const} = f(0)$$

and

$$F\big|_{1 \times I} = \text{const} = f(1).$$

This homotopy yields a pair of



singular 2-simplices σ_1 & σ_2 in X.

$$\sigma_1|_{[v_0,v_3]} = \sigma_2|_{[v_0,v_3]}$$

$$\partial\sigma_1 = \sigma_1|_{[v_1,v_3]} - \sigma_1|_{[v_0,v_3]} + \sigma_1|_{[v_0,v_1]}$$

$$= \text{const}\,\overline{f(1)} - \sigma_1|_{[v_0,v_3]} + f$$

$$\partial\sigma_2 = \sigma_2|_{[v_2,v_3]} - \sigma_2|_{[v_0,v_3]} + \sigma_2|_{[v_0,v_2]}$$

$$= g - \sigma_2|_{[v_0,v_3]} + \text{const}\,\overline{f(0)}$$

We compute

$$\partial(\sigma_1 - \sigma_2) = \text{const}\,\overline{f(1)} - \cancel{\sigma_1|_{[v_0,v_3]}} + f$$

$$- g + \cancel{\sigma_2|_{[v_0,v_3]}} - \text{const}\,\overline{f(0)}$$

$$= f - g + \text{const}\,\overline{f(1)} - \text{const}\,\overline{f(0)}$$

Since constant singular simplices are

boundaries, so is $f-g$. This implies that $f \underset{H}{\simeq} g$. $\blacksquare$

Now that we have proved these lemmas we return to the proof of the Hurewicz theorem.

First we need a map from $\pi_1(X,x_0)$ to $H_1(X)$:

$$\phi : \pi_1(X,x_0) \to H_1(X)$$

Let $\{f\} \in \pi_1(X,x_0)$ and let $f: I \to X$ be a loop representing $\{f\}$ in G.

f is a cycle since

$$\partial f = f(1) - f(0) = x_0 - x_0 = 0.$$

Define

$$\phi(\{f\}) := [f].$$

CLAIM: ϕ is well defined

This statement follows from Lemma 3.

Let $g \in \{f\}$. Then $f \underset{\pi}{\simeq} g$ by definition.

By lemma 3 we also have that

$f \underset{H}{\simeq} g$, ie. $[f] = [g]$.

CLAIM: ϕ is a homomorphism of groups.

Let $f, g : I \to X$ be two loops based at x_0. Then

$$\phi(\{f\} * \{g\}) = \phi(\{f * g\})$$

$$= [f * g] \overset{*}{=} [f] + [g] = \phi(\{f\}) + \phi(\{g\})$$

*By Lemma 1 $[f * g] = [f] + [g]$.

Since $H_1(X)$ is abelian,
ϕ sends $[G,G]$ to 0.

$\Rightarrow \phi$ induces a homomorphism

$$\phi_* : G^{ab} \to H_1(X).$$

THEOREM [HUREWICZ]

ϕ_* is an isomorphism of groups.

Proof

For all $x \in X$, choose in an arbitrary way a path λ_x from x_0 to x in such a way that $\lambda_{x_0} = \text{const}$.

Define a homomorphism

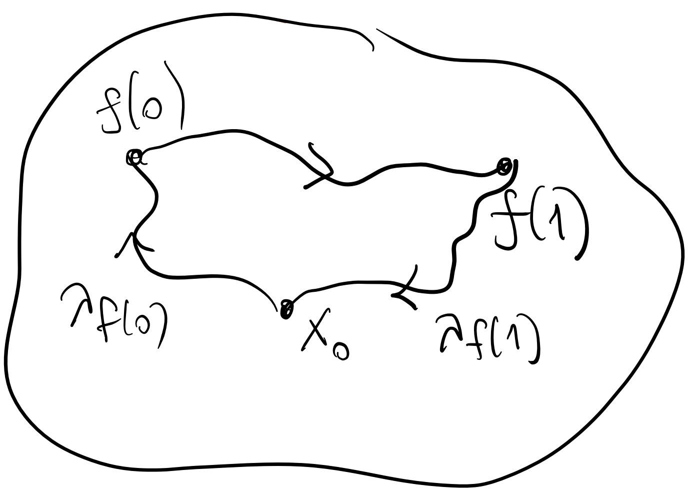
$$\Psi : S_1(X) \to G^{ab}$$

as follows:

for a generator of $S_1(X)$

$f: I \to X$, put

$$\Psi(f) = \{ \lambda_{f(0)} * f * \lambda_{f(1)}^{-1} \} \in G^{ab}$$



(I is identified with Δ^1)

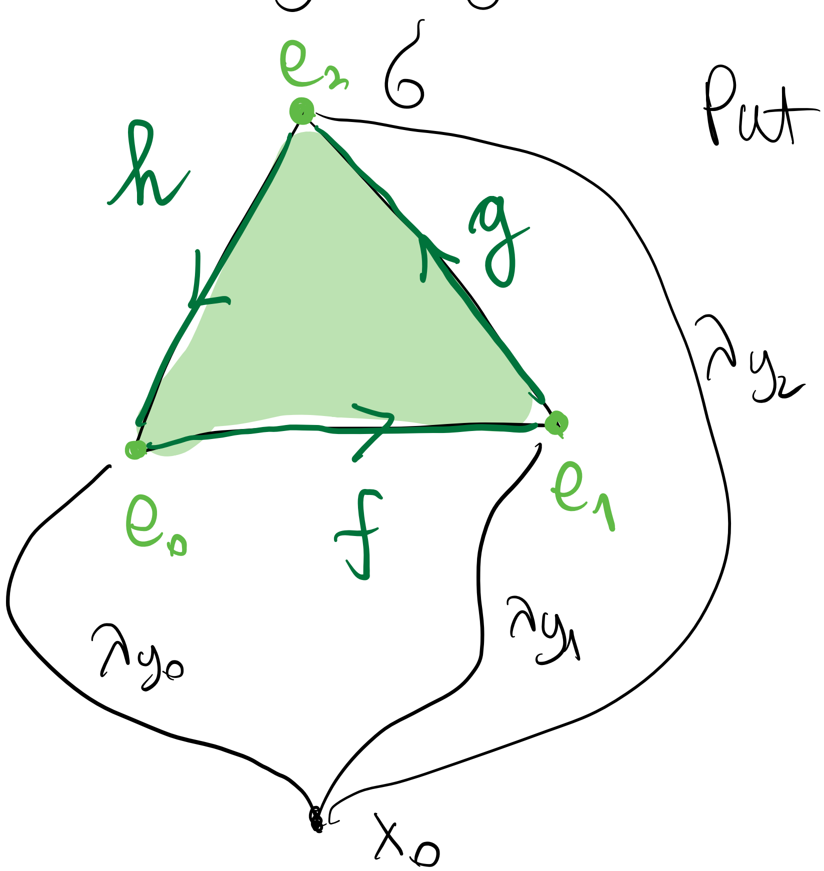
As $S_1(X)$ is free-abelian & G^{ab} is abelian, the above defines uniquely the homomorphism Ψ.

Lemma 4

$\forall b \in B_1(X)$ we have $\Psi(b) = 1 \in G^{ab}$.

Proof

Because Ψ is a homomorphism, it is enough to check that Ψ gets the value 1 on b's of the type $b = \partial\sigma$, where $\sigma: \Delta^2 \to X$ is any singular 2-simplex.



Put $y_i = \sigma(e_i)$

$f := \sigma|_{e_0 e_1}$

$g := \sigma|_{e_1 e_2}$

$h := \sigma|_{e_2 e_0}$

$$\Psi(\partial G) = \Psi(g - h^{-1} + f) =$$

$$= \Psi(g) * (\Psi(h^{-1}))^{-1} * \Psi(f)$$

G^{ab} abelian

$$= \Psi(f) * \Psi(g) * (\Psi(h^{-1}))^{-1}$$

$$= \{\lambda_{y_0} * f * \underbrace{\lambda_{y_1}^{-1} * \lambda_{y_1}}_{\text{homotopic to const rel } \partial I} * g * \lambda_{y_2}^{-1} * (\lambda_{y_0} * h^{-1} * \lambda_{y_2}^{-1})^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * \underbrace{\lambda_{y_2}^{-1} * \lambda_{y_2}}_{\underset{\pi}{\simeq} \text{ const rel } \partial I} * h * \lambda_{y_0}^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * h * \lambda_{y_0}^{-1}\} = (*)$$

Now $f * g * h \underset{\pi}{\simeq} \text{const}_{y_0} \text{ rel } \partial I$

$$(*) = \{\lambda_{y_0} * \lambda_{y_0}^{-1}\} = 1 \in G^{ab}$$

So far we have

ϕ, ϕ_*, Ψ.

Since $\Psi(B_1(X)) = \{1\}$, Ψ induces a homomorphism

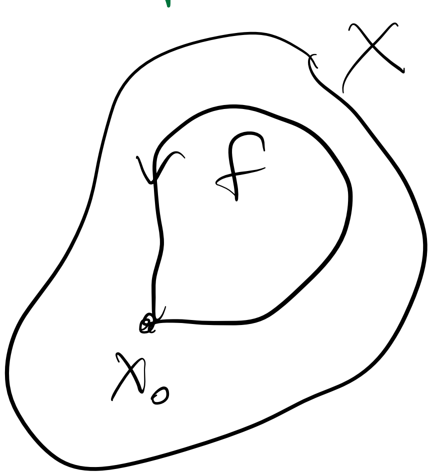
$$\Psi_* : H_1(X) \to G^{ab}.$$

(we restrict Ψ to Z_1)

CLAIM

$\Psi_* \circ \phi_* = id$

Proof



If f is a loop based in x_0, then