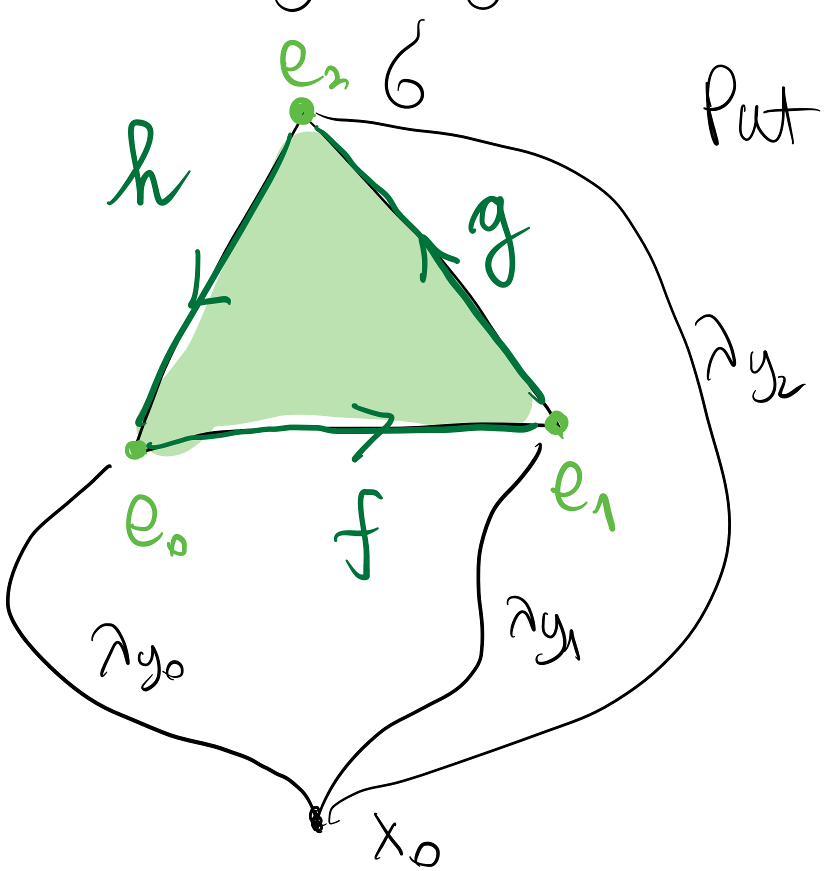
Lemma 4

$\forall b \in B_1(X)$ we have $\Psi(b) = 1 \in G^{ab}$.

Proof

Because Ψ is a homomorphism, it is enough to check that Ψ gets the value 1 on b's of the type $b = \partial\sigma$, where $\sigma: \Delta^2 \to X$ is any singular 2-simplex.



Put $y_i = \sigma(e_i)$

$f := \sigma|_{e_0 e_1}$

$g := \sigma|_{e_1 e_2}$

$h := \sigma|_{e_2 e_0}$

$$\Psi(\partial\sigma) = \Psi(g - h^{-1} + f) =$$

$$= \Psi(g) * (\Psi(h^{-1}))^{-1} * \Psi(f)$$

G^{ab} abelian

$$= \Psi(f) * \Psi(g) * (\Psi(h^{-1}))^{-1}$$

$$= \{\lambda_{y_0} * f * \underbrace{\lambda_{y_1}^{-1} * \lambda_{y_1}}_{\substack{\text{homotopic}\\ \text{to const}\\ \text{rel } \partial I}} * g * \lambda_{y_2}^{-1} * (\lambda_{y_0} * h^{-1} * \lambda_{y_2}^{-1})^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * \underbrace{\lambda_{y_2}^{-1} * \lambda_{y_2}}_{\substack{\simeq_{\partial I} \text{ const}\\ \text{rel } \partial I}} * h * \lambda_{y_0}^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * h * \lambda_{y_0}^{-1}\} = (*)$$

Now $f * g * h \underset{\pi}{\simeq} \text{const}_{y_0}$ rel ∂I

$$(*) = \{\lambda_{y_0} * \lambda_{y_0}^{-1}\} = 1 \in G^{ab}$$

So far we have

ϕ, ϕ_*, Ψ.

Since $\Psi(B_1(X)) = \{1\}$, Ψ

induces a homomorphism

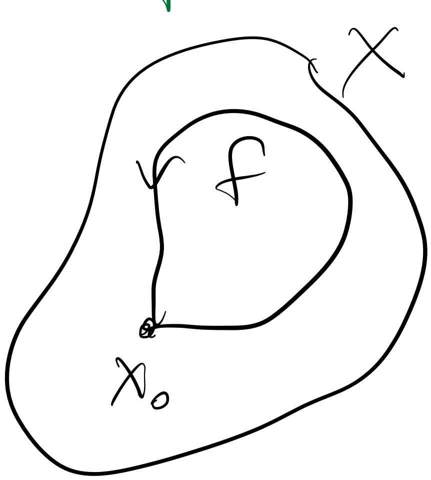
$$\Psi_* : H_1(X) \to G^{ab}.$$

(we restrict Ψ to Z_1)

CLAIM

$\Psi_* \circ \phi_* = id$

Proof



If f is a loop based in x_0, then

$$\Psi_* \circ \phi_*(\{f\}) = \Psi_*([f]) =$$

$$= \{\underset{\uparrow\,\text{const}}{\lambda_{x_0}} * f * \underset{\uparrow\,\text{const}}{\lambda_{x_0}^{-1}}\} = \{f\}$$

CLAIM

$$\phi_* \circ \Psi_* = \mathrm{id}$$

Proof

Note that $X \ni x \longmapsto \lambda_x$ induces a homomorphism

$$S_0(X) \longrightarrow S_1(X)$$

$$\begin{array}{ccc} \cup & & \\ c & \longmapsto & \lambda_c \end{array}$$

$$\sum_{x \in X} n_x x \longmapsto \sum_{x \in X} n_x \lambda_x$$

We will denote $\sum_{x \in X} n_x \lambda_x$ by

$\lambda_{\sum_{x\in X} n_x x}$.

Lemma 5

① Let $\sigma : I \to X$ be a 1-simplex (a path). Then

$$\phi_* \Psi(\sigma) = [\sigma + \lambda_{\sigma(0)} - \lambda_{\sigma(1)}] = $$
$$= [\sigma - \lambda_{\partial\sigma}].$$

② If c is a 1-chain in X, then $\phi_* \Psi(c) = [c - \lambda_{\partial c}]$.

In particular, if c is a cycle, then $\phi_* \Psi(c) = [c]$.

Proof

(1) $\phi_* \Psi(\sigma) = \phi_*\{\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}\} =$

use Lemmas 1 & 2

$= [\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}] =$

$= [\lambda_{\sigma(0)} + \sigma - \lambda_{\sigma(1)}] = [\sigma - \lambda_{\partial\sigma}]$

(2) Follows from the linearity of the map $S_0(X) \ni c \mapsto \lambda_c \in S_1(X)$ and other maps involved here.

If c is a cycle, $\partial c = 0$ &

$\phi_* \Psi(c) = [c - \lambda_{\partial c}] = [c - 0] = [c]$

COROLLARY

$\phi_* \Psi_* [c] = [c]$, ie $\phi_* \circ \Psi_* = id.$

This statement therefore completes the proof.

The next important property of singular homology is homotopy invariance.

HOMOTOPY INVARIANCE

Recall that a continuous map $f: X \to Y$ induces a chain map $f_c: S_\bullet(X) \to S_\bullet(Y)$ between chain complexes $S_\bullet(X)$ and $S_\bullet(Y)$ and f_c in turn induces a map $f_*: H_n(X) \to H_n(Y)$. We have already proved that if f is a homeomorphism, then f_* is an isomorphism. Now we turn our attention to maps between homology groups induced by homotopic maps. In particular, we can prove the following theorem [For this part we follow Hatcher]

THEOREM

If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \to H_n(Y)$$

In particular, if f is a homotopy equivalence, then f_* is an isomorphism for all n.

Proof

The essential ingredient of the proof is to subdivide $\Delta^n \times I$ into simplices.

For a general n:

Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$

and $\Delta^1 \times \{1\} = [w_0, \dots, w_1]$,

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$.

We pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n-simplices each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0.

First step: $[v_0, \dots, v_n] \to [v_0, \dots, v_{n-1}, w_n]$

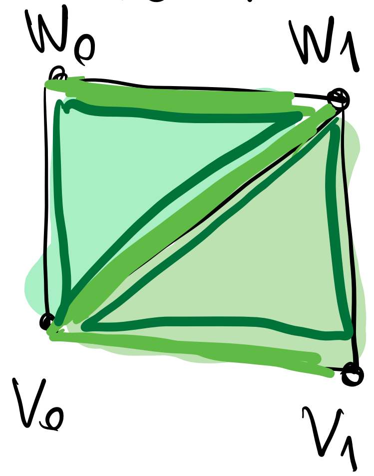
Second step: $[v_0, \dots, v_{n-1}, w_n] \to [v_0, \dots, w_{n-1}, w_n]$

$\vdots$

$[v_0, \dots, v_i, w_{i+1}, \dots, w_n] \to [v_0, \dots, v_{i-1}, w_i, \dots, w_n]$

The region between these two simplices is exactly the $(n+1)$-sx $[v_0, \dots, v_i, w_i, \dots, w_n]$ which has $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ as a lower face and $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ as an upper face.

$n = 1$

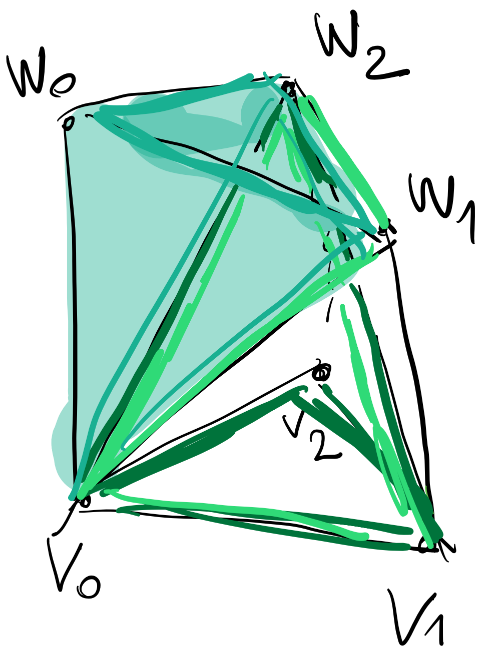


$$\left.\begin{array}{l}[v_0, v_1] \to \\ [v_0, w_1] \to \\ [w_0, w_1]\end{array}\right\}$$ sequence of 1-simplices

Regions in between that are

2-simplices : $[v_0, v_1, w_1]$, $[v_0, w_0, w_1]$

$n = 2$



$[v_0, v_1, v_2] \rightarrow$

$[v_0, v_1, w_2] \rightarrow$

$[v_0, w_1, w_2] \rightarrow$

$[w_0, w_1, w_2]$

Regions in between that are

3-simplices : $[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Altogether, $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an n-simplex face.

Given a homotopy $F: X \times I \to Y$ from f to g we define

$$P_n : S_n(X) \to S_{n+1}(Y),$$

a homomorphism of groups given on generators by the following formula:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times id_I)\Big|_{[v_0, \dots, v_j, w_j, \dots, w_n]}}_{\text{these are singular } (n+1)\text{-simplices}}$$