HYPERBOLIC GEOMETRY

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1. Hyperbolic Geometry

Hyperbolic geometry also sometimes known as Bolyai–Lobachevskian Geometry came to be when it was discovered that Euclid's 5th postulate is independent of the other axioms. So by changing the 5th postulate one can get axiomatic spherical and hyperbolic geometry on top of the regular euclidean geometry. In short, if we assume a straight line has exactly one parallel we get euclidean geometry. If we assume there are no parallel lines to a given line, we get spherical geometry. And finally if we assume there is at least two parallel lines we get hyperbolic geometry. This leads to the fact that in hyperbolic geometry parallel lines diverge and triangles have interior angles strictly bigger than 180 deg. Intuitively one can imagine that there is more space than there *should* be.

It is interesting to note that hyperbolic geometry is entirely consistent with euclidean geometry. That means that hyperbolic geometry can be modeled by euclidean geometry and vice versa. There are a lot of similarities between hyperbolic and euclidean geometries but there are also many differences.

Hyperbolic Geometry exists in any dimension, but we will be discussing the hyperbolic plane in particular.

2. Models of the hyperbolic plane

There are a multitude of models of the hyperbolic plane, but we will focus on two of them.

2.1. Poincare disk model.

In the Poincaré disk model the hyperbolic plane is represented by a disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and distances are bigger closer to the edge of the disk. We will formalize this notion later. Note that the border is excluded. Points in the disk correspond to points in the hyperbolic plane. Straight lines are represented by circular arcs that intersect the circumference $\partial\mathbb{D}$ at right angles or by diameters of the disk. Circles are represented by circles contained in $\mathbb{D}$, where the hyperbolic center point is closer to the center of the disk than the euclidean center point.

The set of ideal points is exactly the perimeter of the disk $\partial\mathbb{D}$.

2.2. Poincaré half plane model.

For the half plane model we take the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ where similarly the distance is bigger closer to the real axis. Straight lines are vertical lines or similarly to above semicircles with that intersect the real axis at right angles, so semicircles with their center on the real axis. Hyperbolic circles are euclidean circles entirely contained in $\mathbb{H}$, where again the hyperbolic center is closer to the bottom than the euclidean center.

The set of ideal points in this model is the real axis as well as a single point at $y = +\infty$.

Both of these models are conformal, so angles in the model correspond to angles in the hyperbolic plane.

There is a simple map from one model to the other given by $f : \mathbb{D} \to \mathbb{H}, z \mapsto \frac{z-i}{z+i}$

We will be using the half plane model almost exclusively for computation as it is easier to deal with, but the disk model is very useful for intuition and visualization.

3. Riemannian Metric

To formalize the notions stated above we will need to introduce some differential geometry.

Definition 3.1. The *tangent bundle* of the hyperbolic plane is defined as $T\mathbb{H} = \mathbb{H} \times \mathbb{C}$. The tangent bundle at a point $z \in \mathbb{C}$ is then defined as $T_z\mathbb{H} = \{z\} \times \mathbb{C}$.

This is the space in which derivatives live. One can image the tangent bundle at a pint as a plane touching the hyperbolic plane in only one point. Note that we will understand $T_z\mathbb{H}$ as a 2 dimensional vector space over $\mathbb{R}$ in the second variable. For any (z, v) and (z, w) in $T_z\mathbb{H}$ define their scalar product as follows

$$\langle v, w \rangle_z = \frac{1}{\Im(z)}(v, w)$$

where (v, w) is the standard 2 dimensional scalar product.

Definition 3.2. A *path* $\phi : [0, 1] \to \mathbb{H}$ is a continuous, piece wise differentiable curve.

The length of a path ϕ is defined as follows:

$$L(\phi) = \int_0^1 \|D\phi(t)\|_{\phi(t)} dt.$$

Where $D\phi(t) \in T_{\phi(t)}\mathbb{H}$ is the tangent vector $(\phi(t), \phi'(t))$.

In practice one can use the rule of thumb $dh = \frac{1}{\Im(z)} \cdot dz$ for calculating path integrals.

We can now define the distance between two points.

Definition 3.3. The *distance* between two points $z_1, z_2 \in \mathbb{H}$ is the infimum of the length of path between them.

$$d(z_1, z_2) = \inf_{z_1 \overset{\phi}{\rightsquigarrow} z_2} L(\phi)$$

Note that this distance is indeed a metric. This follows quickly from the definition.

4. Circle circumference

All the following calculations are done in $\mathbb{H}$. Let's observe a (euclidean) circle with radius $0 < r_e < 1$ and center at $i \in \mathbb{H}$. We can parameterize it with $\phi : [0, 1] \to \mathbb{H}, t \mapsto 1 + \exp(t \cdot 2\pi i)$. We observe $\phi'(t) = r \cdot 2\pi i \exp(t \cdot 2\pi i)$. Note that this euclidean circle corresponds to a hyperbolic circle with different radius. Now let us calculate the hyperbolic circumference. $L(\phi)$.

$$\begin{aligned} L(\phi) &= \int_0^1 \|D\phi(t)\|_{\phi(t)} dt \\ &= \int_0^1 \sqrt{\frac{1}{\Im(\phi(t))^2} \cdot (\phi'(t), \phi'(t))}\, dt \\ &= \int_0^1 \frac{1}{\Im(\phi(t))} \cdot \|\phi'(t)\|_{\mathbb{R}^2}\, dt \\ &= \int_0^1 \frac{1}{1 + sin(t \cdot 2\pi)} \cdot 2\pi r_e\, dt \\ &= 2\pi r_e \cdot \frac{1}{\sqrt{1 - r_e^2}} \end{aligned}$$

Note that the last step is a tricky real integral that we will not discuss further as it is outside the scope.

5. Isomorphisms of the hyperbolic plane

In this section, we will find some isomorphisms of the hyperbolic plane, i.e. bijections from the hyperbolic plane to itself that preserve the Riemannian metric.

Specifically, we will introduce a class of functions called *Möbius Transformations*. It will be useful to define them as the group action on something called *projective spaces.*

5.1. **Projective spaces.**

A projective space is a type of quotient space. Intuitively, we obtain a projective space if we "ignore" scalar multiples.

5.1.1. *The projective complex line.*

Definition 5.1. The *projective complex line* is the quotient space

$$\mathrm{P}\mathbb{C} = \mathbb{C}^2 \setminus \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\} \big/ \sim$$

where for any two $v, w \in \mathbb{C}^2$, the equivalence relation $v \sim w$ holds if and only if there is some nonzero scalar $\lambda \in \mathbb{C}^\times$ such that $v = \lambda w$. We denote the equivalence class containing all scalar multiples of $v \in \mathbb{C}^2 \setminus \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\}$ by $[v] \in \mathrm{P}\mathbb{C}$. Whenever it is clear from the context, we may omit the brackets and just write $v \in \mathrm{P}\mathbb{C}$.

5.1.2. *Projective matrix groups.*

The projective matrix groups can be constructed in a similar way as the quotient of matrix groups such as $\mathrm{SL}_2(\mathbb{C})$ with some equivalence relation. However, we will instead take a group quotient, which gives the resulting space additional structure.

Definition 5.2. The *complex projective special linear group (of dimension two)* is the quotient group

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \big/ \{\pm \mathbb{1}\}.$$

We denote the coset containing $M \in \mathrm{SL}_2(\mathbb{C})$ by $[M] \in \mathrm{PSL}_2(\mathbb{C})$. Again, we may just write $M \in \mathrm{PSL}_2(\mathbb{C})$ if it is clear from context.

We also define the *real projective special linear group*

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \big/ \{\pm \mathbb{1}\}$$

and the *projective orthogonal group*

$$\mathrm{PSO}(2) = {}^{\mathrm{SO}(2)}\big/_{\{\pm\mathbb{1}\}}.$$

Because $\{\pm\mathbb{1}\}$ is a normal subgroup of those matrix spaces (i.e. $\{\pm\mathbb{1}\} \triangleleft SL_2(\mathbb{C})$ etc.), these quotient spaces are groups with the well-defined operation $[M][N] = [MN]$.

5.1.3. *The action of* $\mathrm{PSL}_2(\mathbb{C})$ *on* $\mathrm{P}\mathbb{C}$. The group $\mathrm{PSL}_2(\mathbb{C})$ and its subgroups act on $\mathrm{P}\mathbb{C}$ by $[M][v] = [Mv]$. In words, we take the matrix-vector product of the representatives M and v of the equivalence classes $[M]$ and $[v]$.

5.2. **Möbius Transformations.** We will now show a way to identify $\mathrm{P}\mathbb{C}$ with the extended complex plane $\hat{\mathbb{C}}$. For any $z \in \mathbb{C}$, we identify $z \in \hat{\mathbb{C}}$ with $\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$. We also identify $\infty \in \hat{\mathbb{C}}$ with $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$. It can be shown that each element of $\mathrm{PSL}_2(\mathbb{C})$ is identified with exactly one element of $\hat{\mathbb{C}}$ and vice versa.

Under this identification, the action of some $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$ corresponds to the function

$$\begin{aligned} m_A : \hat{\mathbb{C}} &\to \hat{\mathbb{C}} \\ z &\mapsto \begin{cases} \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty \\ \frac{az+b}{cz+d} & \text{otherwise} \end{cases} \end{aligned}$$

with $ad - bc = 1$.

Definition 5.3. A function of the above form is called a *Möbius transformation.*

Note. It is sometimes useful to replace the condition $ad-bc = 1$ with the condition $ad-bc \neq 0$. This (seemingly weaker) condition is actually equivalent in the context of Möbius transformations: For any function $z \mapsto \frac{az+b}{cz+d}$ that satisfies the "weak" condition $ad - bc \neq 0$, we can rewrite it as

$$z \mapsto \frac{az+b}{cz+d} = \frac{u}{u}\frac{az+b}{cz+d} = \frac{\frac{a}{u}z + \frac{b}{u}}{\frac{c}{u}z + \frac{d}{u}}.$$

The right-hand side satisfies the (seemingly stronger) condition $\frac{a}{u}\frac{d}{u} - \frac{b}{u}\frac{c}{u} = 1$, therefore the function – no matter how it is written – is a Möbius transformation.

For example, the function $z \mapsto \frac{z-i}{z+i}$ which maps the disk model $\mathbb{D}$ to the half-plane model $\mathbb{H}$ is a Möbius transformation, even though $ad - bc = 1 \cdot i - (-i) \cdot 1 = 2i \neq 1$.

5.3. **The real procetive special linear group** $\mathrm{PSL}_2(\mathbb{R})$**.** Since we are interested in isomorphisms of the hyperbolic plane (as modelled by the upper half-plane), we need to restrict ourselves to Möbius transformations that map the upper half-plane to itself.

Lemma 5.4. *Möbius transformations that correspond to elements of* $\mathrm{PSL}_2(\mathbb{R})$*, i.e. Möbius transformations* $z \mapsto \frac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad-bc = 1$*, map the upper half-plane to itself.*

We will prove this after the next lemma. The converse is also true, i.e. any Möbius transformation that maps the upper half-plane to itself corresponds to an element of $\mathrm{PSL}_2(\mathbb{R})$, but we will not prove this because we will not use it later.

Note. Contrary to the last note, we *cannot* replace $ad - bc = 1$ with $ad - bc \neq 0$. For example, the function $z \mapsto \frac{1}{z}$, when normalized such that $ad - bc = 1$, equals $z \mapsto \frac{i}{zi}$, which does not have real coefficients, and therefore is not an element of $\mathrm{PSL}_2(\mathbb{R})$. We can, however, replace $ad - bc = 1$ with $ad - bc > 0$.

Lemma 5.5. *Let $f : z \mapsto \frac{az+b}{cz+d}$ be a Möbius transformation in $\mathrm{PSL}_2(\mathbb{R})$, i.e. with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Then it holds that*

$$\Im(f(z)) = \frac{\Im(z)}{|cz + d|^2} \tag{5.6}$$

Proof. We set $z =: x + iy$ and calculate

$$\begin{aligned}
\Im\left(f(z)\right) &= \Im\left(\frac{az+b}{cz+d}\right) = \Im\left(\frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}}\right) \\
&= \Im\left(\frac{(ax+aiy+b)(cx-ciy+d)}{|cz+d|^2}\right) \\
&= \Im\left(\frac{acx^2 - acixy + adx + acixy + acy^2 + adiy + bcx - bciy + bd}{\underbrace{|cz+d|^2}_{\in\mathbb{R}}}\right) \\
&= \frac{\Im(acx^2 - acixy + adx + acixy + acy^2 + adiy + bcx - bciy + bd)}{|cz+d|^2} \\
&= \frac{-acxy + acxy + ady - bcy}{|cz+d|^2} \\
&= \frac{\overbrace{(ad-bc)}^{=1}\, y}{|cz+d|^2} \\
&= \frac{\Im z}{|cz+d|^2}
\end{aligned}$$

$\square$

Proof of Lemma 5.4. By Lemma 5.5 it holds for any z that $\Im(f(z)) = \frac{\Im(z)}{|cz+d|^2}$. In particular, $\Im(f(z))$ is positive if $\Im(z)$ is, meaning that f maps the upper half-plane to itself. $\square$

5.4. $\mathrm{PSL}_2(\mathbb{R})$ **are isomorphisms.** We will now show a main theorem of this talk, namely that elements of $\mathrm{PSL}_2(\mathbb{R})$ are isomorphisms of the hyperbolic plane modelled by the upper half-plane.

Definition 5.7. Let $\mathbb{A}$ and $\mathbb{B}$ be two spaces with corresponding Riemannian metrics $\langle\cdot,\cdot\rangle_a, \langle\cdot,\cdot\rangle_a$. An *isomorphism* is a function $\varphi : A \mapsto B$ that preserves the Riemannian metric, meaning that for any $(a, v), (a, w) \in T_a^1\mathbb{A}$, we have

$$\langle v, w\rangle_a = \langle\varphi'(v), \varphi'(w)\rangle_{\varphi(a)}$$

Theorem 5.8. *The action of every Element of* $\mathrm{PSL}_2(\mathbb{R})$ *is an isomorphism of the hyperbolic plane (as modelled by the upper half-plane) to itself.*

Note. The converse of the above theorem is not true, however $\mathrm{PSL}_2(\mathbb{R})$ does contain all *orientation-preserving* isomorphisms of the hyperbolic plane.

Proof. Let $f : z \mapsto \frac{az+b}{cz+d}$ be a Möbius transformation in $\mathrm{PSL}_2(\mathbb{R})$, and let $(z,v),(z,w) \in T^1_z\mathbb{H}$. We already know from Lemma 5.4 that f maps $\mathbb{H}$ to itself.

It remains to show that f preserves the Riemannian metric. Indeed, we have

$$\begin{aligned}\langle f'(z)v, f'(z)w\rangle_{f(z)} &\overset{\text{def}}{=} \frac{1}{\Im(f(z))^2}\,(f'(z)v, f'(z)w)\\ &\overset{(5.6)}{=} \frac{|cz+d|^4}{\Im(z)^2}\,(f'(z)v, f'(z)v)\\ &= \frac{|cz+d|^4}{\Im(z)^2}\left(\frac{1}{(cz+d)^2}v, \frac{1}{(cz+d)^2}w\right)\\ &\overset{\star}{=} \frac{1}{\Im(z)^2}\,(v,w) \overset{\text{def}}{=} \langle v,w\rangle_z.\end{aligned}$$

Here $f'(z)$ denotes the total derivative of f at z. Equation $\star$ is an application of a property of the euclidean scalar product. This concludes the proof. □

5.5. **Simple transitivity of** $\mathrm{PSL}_2(\mathbb{R})$ **on** $T^1\mathbb{H}$**.** We have already seen the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$. If we consider the movement of unit tangent vectors under those Möbius transformations, we get a related action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^2\mathbb{H}$. For any $f \in \mathrm{PSL}_2(\mathbb{R})$ and $(z,v) \in T^1\mathbb{H}$, it is defined by $Df(z,v) = (f(z), f'(z)v)$.

This action is *transitive*. This means that for any $(y,v),(z,w) \in T^1\mathbb{H}$, there is some $f \in \mathrm{PSL}_2(\mathbb{R})$ such that $Df(y,v) = (z,w)$. Intuitively, this means that hyperbolic space "looks the same" at every point and in every direction.

The action is even *simply transitive*, meaning that there is *exactly* one such f. This allows us to identify $\mathrm{PSL}_2(\mathbb{R})$ with the hyperbolic plane as modelled by the half-plane. For this, we simply choose some base point (z,v) and identify $f \in \mathrm{PSL}_2(\mathbb{R})$ with $Df(z,v)$. The usual choice for a base point is the unit tangent vector pointing in the direction of the positive imaginary axis at $z = i$.

6. Some objects in hyperbolic space

The most basic kind of objects in hyperbolic space are points. As we have seen in section 2, there are also *ideal* points, which are "points" at infinity. In this section we will introduce some more of the basic geometric shapes and show, without proof, some of their properties.

6.1. **Geodesics.** Geodesics are the generalization of lines in Euclidean space.

Definition 6.1. A *geodesic segment* is the path from some point a to some point b which is the shortest among all paths between a and b.

A *geodesic line* or just *geodesic* is a path whose endpoints are ideal points such that for any two points a and b on the geodesic line, the geodesic segment from a to b is contained in the geodesic line.

We have already mentioned in chapter 2 that the geodesics in the half plane model are vertical lines and the upper halves of circles that meet the real axis at a right angle. Like in Euclidean space, there is exactly one geodesic segment between and exactly one geodesic line through any two points. Unlike Euclidean geometry, however, a line can have many parallels through the same point. There is also a distinction between *limiting parallels* which meet at an ideal point and *ultraparallels* which don't.

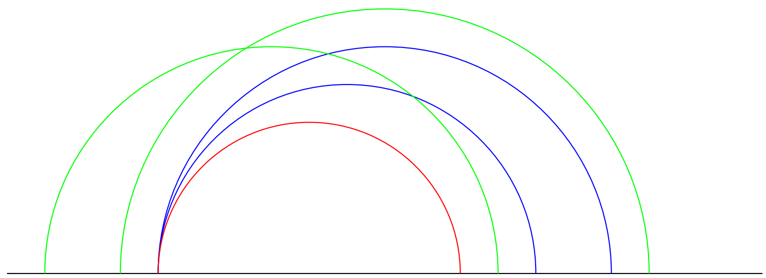

FIGURE 1. A geodesic (red), two limiting parallels (blue) and two ultraparallels through the same point (green)

6.2. **The geodesic flow.** The matrix group

$$A = \left\{ a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

called the *geodesic flow*, gives a way of representing geodesics. In particular, if $g \in \mathrm{PSL}_2(\mathbb{R})$ is a point of the hyperbolic plane (remember our identification of $\mathrm{PSL}_2(\mathbb{R})$ with $T^1\mathbb{H}$), then $Ag = \{a_t g \mid t \in \mathbb{R}\}$ is (kind of) a geodesic of hyperbolic space: It is not *quite* a geodesic of $\mathbb{H}$ since it is a subset of $\mathrm{PSL}_2(\mathbb{R})$, which we identified with the unit tangent space $T^1\mathbb{H}$. The base points of the unit vectors in Ag form a geodesic, and their direction vectors point in the direction of the geodesic. If $g \in \mathrm{PSL}_2(\mathbb{R})$ corresponds to $(z, v) \in T^1H$, then this geodesic is the one that passes through z in the direction of v.

6.3. **Circles, horocycles and hypercycles.** A circle, like in Euclidean geometry, is the set of points at a constant distance from a center.

Lemma 6.2. *In both the disk and the half-plane models, hyperbolic circles are modelled by (Euclidean) circles.*

Proof sketch. The easiest proof uses the disk model, on which we haven't defined the hyperbolic metric yet. Because of that, we will only give a sketch of the proof.

It can be shown by the symmetry of the disk model that any hyperbolic circle whose center lies in the center of the disk must be modelled by an Euclidean circle. We can then move the hyperbolic plane to get the center to any other point using a Möbius transformation, and since Möbius transformations map circles and lines to circles and lines, we can show that the translated hyperbolic circle is still modelled by an euclidean circle (it cannot be modelled by a line since that would intersect the boundary of the model). For the case of the half-plane model, we can use the same argument since the map that identifies the two models is also a Möbius transformation. □

Unlike Euclidean space, hyperbolic space has "circles with infinite radius" (that are different from lines). A *horocycle* in the hyperbolic plane is a circle in the disk or half-plane model that touches the model boundary. In the half-plane model, they can also be modelled by a horizontal line (which touches the boundary at infinity).

If, in either model, you take a circle or a line that intersects the model boundary twice, then the part of this circle or line that lies inside the model is called a *hypercycle* (unless it is a geodesic). This is the second kind of "circle with infinite width".

6.4. **The horocycle flow.** The *horocycle flow* is a similar construction to the geodesic flow. It uses the matrix group

$$U = \left\{ u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}.$$

Its orbits Uz correspond to horocycles in the same sense as the orbits of the geodesic flow correspond to geodesics. The unit vectors are perpendicular to the circle, pointing towards the outside.