1 Linear Groups

1.1 Basics

Definition 1. *A group G is linear if it can be embedded into the matrix group* $\mathrm{GL}(n, K)$ *(for some field K, typically $K = \mathbb{R}, \mathbb{C}$) such that this image is closed under the natural topology on* $\mathrm{GL}(n, K) \cong K^{n^2}$. *Such a group (sometimes without the requirement that it is closed in* $\mathrm{GL}(n, K)$*) is sometimes called a* matrix Lie group.

Linear groups a special case of *Lie groups*[1]; a field of study in their own right, which is central in many parts of mathematics, e.g. differential geometry and group theory, as well as in theoretical physics, where they provide an algebraic way to study the symmetries of a given physical system. At appropriate points, we mention how the generalization to Lie groups works, but it will not be a point of focus.

Some examples of linear groups are:

- $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(\{1\})$, (note that $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is continuous)
- $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, as well as their complex counterparts $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ and
- $\mathrm{PSL}(2, \mathbb{R})$, while not embeddable into $\mathrm{GL}(2, \mathbb{R})$, can actually be embedded into $\mathrm{GL}(4, \mathbb{R})$ (see [EW10], section 9.3).

The groups in the first two points are some of the so called *classical groups*.

1.2 The Exponential Map

We will want to define a Riemannian metric on a linear group G. To this end, we shall need to identify the tangent spaces $T_g G$. For now, we will concentrate on the tangent space at the identity, $T_e G$, and hope to exploit the group structure to study $T_g G$ for any $g \in G$.

Let us first consider the example of $G = \mathrm{SO}(2, \mathbb{R})$, the group of rotations of the plane. Geometrically, it is intuitive that this has the same group and topological structure as $S^1 \subseteq \mathbb{C}$. If we write

$$\mathrm{SO}(2, \mathbb{R}) = \{R(\alpha) \mid \alpha \in [0, 2\pi)\} \quad \text{where} \quad R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

then the isomorphism between S^1 and $\mathrm{SO}(2, \mathbb{R})$ is given by $e^{i\alpha} \mapsto R(\alpha)$. In S^1, it is immediate that the tangent space at 1 should is the imaginary line $i\mathbb{R}$. Indeed, any path $\phi : [a, b] \to S^1$ is of the form $t \mapsto e^{i\alpha(t)}$ for some $\alpha : [a, b] \to \mathbb{R}$, and differentiating gets us

$$\frac{d}{dt}\phi = i\alpha'(t)\phi(t) \quad \text{such that} \quad \phi'(t_0) = i\alpha'(t_0) \in i\mathbb{R} \quad \text{for any } t_0 \in [a, b] \text{ with } \phi(t_0) = 1. \tag{1}$$

By considering the paths given by $\alpha(t) := rt$ for $r \in \mathbb{R}$, we have formally shown that the tangent space at 1 is $i\mathbb{R}$. Note that the map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ sends $i\mathbb{R}$ onto the circle S^1 (while trivial, the fact will become much more interesting when we generalize this example).

Even though this result was clear from the getgo, it allows us to think about the same question in $\mathrm{SO}(2, \mathbb{R})$. Begin by noting that $i = e^{i\pi/2}$ corresponds to the rotation

$$R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

[1] after Sophus Lie, Norwegian mathematician of the 19th century

and since $e^{i\alpha} = 1 + i\alpha - \alpha^2/2 + ... \in S^1$, we may come up with the idea to try the same thing in $\mathrm{SO}(2, \mathbb{R})$:

$$I + \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\alpha^2}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 + \dots .$$

A priori, it is not obvious that series converges and leaves us with a matrix in $\mathrm{GL}(2, \mathbb{R})$, but we can perform the calculation and see that

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \quad \text{has} \quad A^{2n} = \alpha^{2n} \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}, \quad A^{2n+1} = \alpha^{2n+1} \begin{pmatrix} 0 & (-1)^{n+1} \\ (-1)^n & 0 \end{pmatrix}.$$

By looking at each component of the matrix individually, we can recognize the structure of the sine and cosine power series, such that

$$R = \exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

which is what we expected because of the isomorphism with S^1. If now $\phi(t) := \exp(tA) = R(t\alpha)$ is a path in $\mathrm{SO}(2, \mathbb{R})$, we can calculate

$$\frac{d}{dt}\phi = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n = A \exp(tA) = \exp(tA)A. \tag{2}$$

which, if $\phi(t_0) = I$, shows $\phi'(t_0) = A$, such that the tangent space at I in $\mathrm{SO}(2, \mathbb{R})$ is $\{\alpha R(\pi/2) \mid \alpha \in \mathbb{R}\}$. Before generalizing our findings, we first seek to gain insight into the matrix exponential.

Definition 2. *The map* $\exp : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\exp(A) := \sum_{k=0}^{\infty} A^k/k!$ *is called the* matrix exponential.

Proposition 1. *The map* exp *is well defined, and absolutely continuous, and the equation* (2) *holds in general.*

1.3 Examples of Matrix Exponentials

Diagonal matrices These are the simplest examples:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{so} \quad A^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix},$$

and it is obvious that $\exp(A) = \mathrm{diag}(e^a, e^b, e^c)$.

Nilpotent matrices Nilpotent matrices (those that are zero to some power) easily lend themselves to this calculation. For example:

$$C = \begin{pmatrix} 0 & b & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C^3 = 0 \quad \Rightarrow \quad \exp(C) = \begin{pmatrix} 1 & b & a + b^2/2 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Upper triangular matrices We examine

$$T = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \quad \text{with} \quad T^2 = \begin{pmatrix} a^2 & * \\ 0 & b^2 \end{pmatrix}, \quad \dots, \quad T^n = \begin{pmatrix} a^n & * \\ 0 & b^n \end{pmatrix}$$

where the entries denoted by $*$ have a somewhat complicated form, but we can see that

$$\exp(T) = \begin{pmatrix} e^a & * \\ 0 & e^b \end{pmatrix}.$$

By the same calculation, the if the diagonal entries of any (upper or lower) triangular matrix A are $a_{11}, \dots, a_{nn}$, then the diagonal of $\exp(A)$ will have entries $e^{a_{11}}, \dots, e^{a_{nn}}$ and thus it's determinant will be given by

$$\det(A) = e^{a_{11}} \cdots e^{a_{nn}} = e^{a_{11} + \dots + a_{nn}} = e^{\operatorname{tr}(A)}.$$

This is the first example of the above formula, which actually holds for all matrices.

Determinant formula By applying Gaussian elimination, any matrix A can be put into a triangular form $T = SAS^{-1}$. We thus have

$$\det(\exp(A)) = \det(\exp(STS^{-1})) = \det(S\exp(T)S^{-1}) = \det(\exp(T)) = e^{\operatorname{tr}(T)} = e^{\operatorname{tr}(STS^{-1})} = e^{\operatorname{tr}(A)}$$

where we have used the simple identity $\exp(STS^{-1}) = S\exp(T)S^{-1}$ which follows from the cancellation $(STS^{-1})^n = ST^nS^{-1}$, as well as the cyclicity of trace $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$. Since e^x is never zero, we see that any matrix in the image of exp is invertible. Having considered sufficient examples, we simply state the following theorem:

Proposition 2. *The map* $\exp : \operatorname{Mat}_{n\times n}(\mathbb{C}) \to \operatorname{GL}(n, \mathbb{C})$ *satisfies*

1. $\exp(0) = I$,
2. $\exp(SAS^{-1}) = S\exp(A)S^{-1}$ *for any* $S \in \operatorname{GL}(n, \mathbb{R})$,
3. $\det(\exp(A)) = e^{\operatorname{tr}(A)}$,
4. *if A and B commute, then* $\exp(A + B) = \exp(A)\exp(B)$ *and so*
5. $\exp(nA) = \exp(A)^n$ *as well as* $\exp(A)^{-1} = \exp(-A)$.

2 Tangent Space at the Identity

Let us consider the example of $G = \operatorname{SO}(2, \mathbb{R})$, the group of 2D rotations. Let once again

$$R(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad A(\alpha) := \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}.$$

Recall what we have found in the introduction: we have shown that $\exp(A(\alpha)) = R(\alpha)$, and can see that the map $\varphi : \mathbb{R} \to G$, mapping $t \mapsto \exp(tA)$ (for any A of the above form) spans the entire group G. Furthermore, at $t = 0$, we have $\varphi(0) = I$ and we know that the derivative with respect to t is $d\varphi/dt = A\exp(tA)$.

At $t = 0$, the derivative is therefore $\exp(0)A = A$, showing that the tangent space at the identity is $\{A(\alpha) \mid \alpha \in \mathbb{R}\}$, just as in the case of the circle in the complex plane. (In fact, this is essentialy the matrix representation of imaginary numbers.)

One parameter subgroup For any linear group, let $\varphi : \mathbb{R} \to G$ be a continuous group homomorphism. Such map φ is called a *one-parameter subgroup*, and it is a theorem ([Hal15], Theorem 2.14) that any such map is of the form $t \mapsto \exp(tA)$ for some $A \in \mathrm{Mat}_{n\times n}(K)$.

As a side note: In the more general case of Lie groups, this doesn't hold anymore, and such one-parameter subgroups may be used to define the (abstract) exponential map.

2.1 The Lie algebra

We have seen that we can use the exponential map on certain matrices to describe the tangent space at the identity in G. The following theorem makes this idea precise, and provides some information about the existence of an inverse map (the logarithm).

Theorem 1. *For any linear group $G \subseteq \mathrm{GL}(n, \mathbb{R})$ there is a neighborhood B of the identity $I \in G$ such that $\log(B) \subseteq \mathrm{Mat}_{n\times n}(\mathbb{R})$ is a neighborhood of zero inside a linear subspace $\mathfrak{g} \subset \mathrm{Mat}_{n\times n}(\mathbb{R})$ which may be characterized via*

1. *$\mathfrak{g}$ is the maximal linear subspace such that $\exp(\mathfrak{g}) \subset G$, or similarly*
2. *$\mathfrak{g}$ is the set of all matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$, or*
3. *$\mathfrak{g}$ consists of all derivatives $\phi'(t)$ of paths $\phi : [a, b] \to G$ at points $t \in [a, b]$ with $\varphi(t) = I$.*

A full proof can be found in [EW10] (Proposition 9.5), and it involves explicity writing the matrix logarithm as a power series.

Proposition 3. *$\mathfrak{g}$ is indeed a vector space, and it has finite dimension.*

Proof. Let $v, w \in \mathfrak{g}$, meaning that there are curves ϕ and ψ such that (after some reparameterization such that the curves are defined over $[-1, 1]$ and are equal to the identity at 0) we have $\phi'(0) = v$ and $\psi'(0) = w$. If we now define $\alpha(t) := \phi(t)\psi(t)$, then $\alpha'(0) = \phi'(0)\psi(0) + \phi(0)\psi'(0) = v + w$, implying $v + w \in \mathfrak{g}$.

For $v \in \mathfrak{g}$ with path $\phi(t)$ and $a \in \mathbb{R}$, it can be immediately seen that the path $t \mapsto a\phi(t)$ implies that $av \in \mathfrak{g}$. Lastly, the zero matrix is contained in $\mathfrak{g}$ because the constant curve $t \mapsto I$ has derivative equal to the zero matrix everywhere.

Lastly, it is clear that $\mathfrak{g}$ is finite-dimensional as a subspace of $\mathbb{R}^{n^2}$. $\square$

Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ We have already identified the Lie algebra of $\mathrm{SO}(2, \mathbb{R})$. As a further example, we consider $G = \mathrm{SL}(n, \mathbb{R})$. To calculate the Lie algebra, note that that any element of $\mathfrak{so}(n, \mathbb{R}) := \mathfrak{g}$ must have trace zero, since $\det(\exp(v)) = e^{\mathrm{tr}(v)} = 1$ if and only if $\mathrm{tr}(v) = 0$. But, by maximality of $\mathfrak{so}(n, \mathbb{R})$, we have

$$\mathfrak{so}(n, \mathbb{R}) = \{v \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \mathrm{tr}(v) = 0\}.$$

The following statement tells us that we can at least partially recover the group G from $\mathfrak{g}$ alone:

Corollary 1. *For any linear group $G \subset \mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g}$ uniquely determines the connected component G^0 of the identity in G. G^0 is generated by $\exp(\mathfrak{g})$ and is an open, closed, path-connected (via smooth curves) and normal subgroup of G.*

We have already seen that a global logarithm cannot exists in our example of $G = \mathrm{SO}(2, \mathbb{R})$, where it is obvious that $A(\alpha) \in \mathfrak{g}$ and $A(\alpha + 2\pi k) \in \mathfrak{g}$ yield the same rotation matrices under exp. However, if we restrict ourselves to the neighborhood $\{A(\alpha) \mid \alpha \in (-\pi, \pi)\}$ of 0, then exp becomes a homeomorphism to $G \setminus R(\pi)$.

A word of warning: In general, $\exp : \mathfrak{g} \to G$ need not even be surjective. Take for example the non-diagonalizable matrix ([Hal15], Example 3.41)

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

If there were a matrix $X \in \mathfrak{g}$ (i.e. a matrix X such that $\mathrm{tr}(X) = 0$) with $\exp(X) = A$, then it also must be non-diagonalizable, otherwise the formula $\exp(S^{-1}XS) = S^{-1}\exp(X)S$ would imply that A is diagonalizable. Thus, X may only have a single eigenvalue, which has to be zero, otherwise $\mathrm{tr}(X) \neq 0$. This means that $Xv = 0$ for some vector $v \in \mathbb{C}^2$, and so $Av = \exp(X)v = \exp(0)v = v$ which is impossible, since A has the single eigenvalue -1.

However, the corollay tells us that A can be written as a product of matrices in the image of exp. Indeed, we can write:

$$\exp(A) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \exp \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \exp \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix}$$

and we are satisfied, since $\mathfrak{so}(2, \mathbb{C})$ consists of traceless 2×2 complex matrices, analogously to its real counterpart.

2.2 Lie Groups and Lie Algebras*

Definition 3. *A* Lie group *is a finite dimensional smooth real manifold that is also a group, and such that multiplication and inversion are smooth maps.*

The theorem 1 has shown us that linear groups can be given local coordinates by using the tangent space $\mathfrak{g}$ (translated, if need be). Furthermore, matrix multiplication and inversion can be written out into a form where it is clear that they are smooth, making linear groups a special case of Lie groups.

Definition 4. *The linear subspace $\mathfrak{g}$ from the above proposition is called the* Lie algebra *and may be considered abstractly as a vector space with a multiplication operation $[\cdot, \cdot]$ satisfying:*

1. *$[\cdot, \cdot]$ is bilinear,*
2. *$[v, v] = 0$ and*
3. *the Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ holds.*

These also imply antisymmetry $[v, w] = -[w, v]$.

In the case of matrix Lie algebras (linear groups are sometimes called matrix Lie groups), the product operation will always be the commutator of two matrices, $[g, h] = gh - hg$.

The above abstract notion of a Lie algebra shall not be important to us, but is noteworthy nontheless. The theory of Lie groups concerns itself with the relation between abstract Lie groups and Lie algebras as above.

2.3 The adjoint representation

For $g \in G$ (note our use of $v, w, \ldots$ for vectors in $\mathfrak{g}$ and $g, h, \ldots$ for elements of the group G) we define

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad v \mapsto g^{-1}vg$$

This map is well defined, since $\exp(tg^{-1}vg) = g^{-1}\exp(tv)g \in G$. In fact, this map is bijective, since $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$, and we may define $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), g \mapsto \mathrm{Ad}_g$. The map Ad is called the *adjoint representation* of G, and it is in fact a *representation* of the group G. Using this map, we can think of elements of G as linear transformations of the Lie algebra $\mathfrak{g}$, and we can gain insight into G by studying Ad.

3 Riemannian Metric

3.1 The Left Translation

Up to this point, we have $T_eG = \{e\} \times \mathfrak{g}$, whose elements consist of (modulo equivalence) of the derivatives $\{\phi'(t_0) \mid \phi : [a,b] \to G, \phi(t_0) = e\}$. If $g \in G$ is any point of G and $\psi : [a,b] \to G$ is a path with $\psi(t_0) = g$, then we can multiply on the left (this is *left translation*) $t \mapsto g^{-1}\psi(t)$, which is the identity at t_0, such that the derivative at the identity, $g^{-1}\psi'(t_0)$, lies in $\mathfrak{g}$.

Similarly, any path at the identity can be sent to one at g, so the tangent spaces T_gG and T_eG are isomorphic, and we can set $T_gG := \{g\} \times \mathfrak{g}$, as well as $TG = G \times \mathfrak{g}$. If $\phi : [a,b] \to G$ is a path differentiable at t_0, we denote the tangent vector

$$D\phi(t_0) := (\phi(t_0), \phi(t_0)^{-1}\phi'(t_0)) \in \{\phi(t_0)\} \times \mathfrak{g} \in TG$$

By definition, we may see that

$$D(g\phi)(t_0) = (g\phi(t_0), v) \qquad D(\phi g^{-1})(t_0) = (\phi(t_0)g^{-1}, gvg^{-1}).$$

We can write the above relations in a slightly different way by considering the left and right translations

$$L_g : G \to G,\ h \mapsto gh \qquad R_g : G \to G,\ h \mapsto hg^{-1}.$$

The derivatives of the translations are then

$$(dL_g)_h : T_hG \to T_{gh}G,\ (h,v) \mapsto (gh, v) \qquad (dR_g)_h : T_hG \to T_{hg^{-1}}G,\ (h,v) \mapsto (hg^{-1}, gvg^{-1})$$

where we have already used the above property of the left translation to identify all tangent spaces.

3.2 The left invariant metric

We have identified the tangent spaces of G and studied translations of them. Since $\mathfrak{g}$ is a finite vector space, we easily choose some inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and define a Riemannian metric on TG by setting $\langle u, v \rangle_g := \langle u, v \rangle$.

Analogously to the construction seen for the hyperbolic plane, this inner product then can be used to define a norm $\|\cdot\|$, the length of curves $L(\phi)$ and finally a metric $d_G(g_0, g_1) := \inf_\phi L(\phi)$ (infimum is over all paths ϕ from g_0 to g_1), where $g_0, g_1 \in G^0$. Note the restriction to the connected component of the identity G^0, which is needed since we can't reach points outside of G^0 via smooth curves. Furthermore, for $h \in G^0$, we have *left-invariance*

$$d_G(L_h g_0, L_h g_1) = d_G(hg_0, hg_1) = d_G(g_0, g_1)$$

which we can see already in the Riemannian metric: if we have $(g, u), (g, v) \in T_g G$, then left translation L_h sends these points (using the induced map dL_h at g) to (hg, u) and (hg, v), but the inner product stays the same: $\langle u, v\rangle_{hg} = \langle u, v\rangle = \langle u, v\rangle_g$. Thus, all objects derived from the Riemannian metric (norm, distance, etc.) are invariant under left translations.

3.3 The induced topology

We shall state, without proof, the following comforting fact:

Proposition 4. *The topology induced on G^0 by the metric d_G is the topology G^0 gets as a subspace of G.*

3.4 Example computation of distance

Once again, let $G = \mathrm{SO}(2, \mathbb{R})$, which has Lie algebra $\mathfrak{g} = \{A(\alpha) \in \mathrm{GL}(2, \mathbb{R}) \mid \alpha \in \mathbb{R}\}$. On $\mathfrak{g}$ we choose a rescaled standard inner product (i.e. considering $\mathfrak{g} \subset \mathbb{R}^4$)

$$\left\langle \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \right\rangle := \alpha\beta$$

A path between $R(0) = I$ and $R(\alpha)$ is given by $\phi : t \mapsto R(t\alpha)$ for $t \in [0, 1]$. Then, the length of this path is

$$L(\phi) = \int_0^1 \|D\phi(t)\|_{\phi(t)} dt, \qquad \|D\phi(t)\|_{\phi(t)} = \sqrt{\langle \phi(t)^{-1}\phi'(t), \phi(t)^{-1}\phi'(t)\rangle}.$$

To carry on the calculation, we recall our earlier formula for deriving the path $t \mapsto \exp(tA(\alpha))$ and calculate

$$\phi'(t) = \frac{d}{dt}\exp(tA(\alpha)) = A(\alpha)\exp(tA(\alpha)) = A(\alpha)R(t\alpha), = R(t\alpha)A(\alpha)$$

such that

$$\|D\phi(t)\|_{\phi(t)} = \sqrt{\langle A(\alpha), A(\alpha)\rangle} = |\alpha|, \quad \text{and} \quad L(\phi) = |\alpha|$$

as expected. This value is an upper bound for the distance $d_G(R(0), R(\alpha)) = \inf_\phi L(\phi)$, but we would have to work a little bit more to actually calculate the distance (geometrically, we might expect it to be proportional to $\min\{|\beta| \mid \exists k : \beta = \alpha + 2\pi i k\}$).

References

[EW10] Manfred Einsiedler and Thomas Ward. *Ergodic Theory*. Springer London, 2010.

[Hal15] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations.* 2nd ed. Springer Cham, 2015.