Primitive Integer Vectors

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Throughout these notes we are following [1] up to and including Proposition 2.

Lets first look at the notation used:

- $G \coloneqq SL_2(\mathbb{R})$
- $\Gamma \coloneqq SL_2(\mathbb{Z})$
- $K \coloneqq SO_2(\mathbb{R})$
- $A \coloneqq \{a_t = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} : t \in \mathbb{R}\}$
- $U \coloneqq \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}$

Note that it is important throughout and when reading [1] that vectors sometimes are interpreted as 1x2 matrices.

Furthermore, we equip $U \cong \mathbb{R}$ with the Lebesque measure as a Haar measure and $K \cong S^1$ with the spherical measure that assigns measure 2π. Note that this measure is induced by

$$\int_0^{2\pi} f(k_\theta)\, d\theta, \text{ where } f \in C(K) \text{ and } k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

We shall not state a Haar measure for G here, one can however be found in [1]. We will however look at the unimodularity of G.

Proposition 1 *G is unimodular, i.e. every right Haar measure is a left Haar measure.*

Proof: Lets assume that m is a right Haar measure on G and fix $g \in G$. Define m_g as $m_g(B) \coloneqq m(gB)$. Now as left multiplication by g is a homeomorphisem, m_g is again a Borel measure with finite measure on compact sets and positive measure on non empty sets. Additionally, it is right invariant:

$$m_g(Bh) = m(gBh) = m(gB) = m_g(B). \tag{2}$$

we can thus conclude that it is again a right Haar measure.
As the Haar measure is unique up to a multiplicative constant, there exists a function $\chi : G \to (0, \infty)$ such that for all $g \in G$ we have $m_g = \chi(g)\, m$. Now note that for any non empty Borel set B we have:

$$\chi(gh)\, m(B) = m_{gh}(B) = m_g(hB) = \chi(g)\, m(hB) = \chi(g)\, m_h(B) = \chi(g)\, \chi(h\chi) \tag{3}$$

and thus $\chi(gh) = \chi(g)\, \chi(h)$.
Now using that $(0, \infty)$ is abelian and some other algebraic arguments found in detail in [1] one can conclude that $\chi \equiv 1$. It follows that $m = m_g$ for all $g \in G$ and thus m is left invariant.□

A quicker way to proof that goes as follows: Note that G is connected simple Lie and thus connected semi simple Lie. Now use the fact that any group that is connected semi simple Lie is unimodular.

Moving on, we will now come to two decompositions and then a result regarding the primitive integer vectors.

Lemma 1 *Iwasawa deomposition*
Let $g \in G$, then there exists unique $k \in K, a \in A, u \in U$ such that $g = uak$

Proof:
Take $v = e_2 g$. Then there exists a unique $k \in K$ such that $vk = ||v|| e_2$. Now choose $t \in \mathbb{R}$ such that $vka_t = e_2$ (Thinking about these properties geometrically can aid the understanding). Now $e_2 ka_t = e_2$ implies $gka_t = u$ for some unique $u \in U$. Rearranging this then yields $g = ua_{-t}k$ for all of those unique.□

Lemma 2 *KAK decomposition*
Let $g \in G$, then there exists a unique t and some $k, l \in K$ such that $g = ka_{\pm}l$. Furthermore, $\forall \epsilon > 0$ it holds that the set

$$V_\epsilon = K\{a_t : |t| < \epsilon\}K \tag{4}$$

is an open neighborhood of the identity.

Proof: See [1] as its rather long.

Proposition 2 *The set of primitive integer vecotrs in $\mathbb{R}^2$ is given by $e_2\Gamma = \Gamma/\Gamma_\infty$, where $\Gamma_\infty := \Gamma \cap U$*

Proof: We shall only prove the first part here. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then ad-bc = 1 which implies that c and d are coprime. Thus $e_2 g = \begin{pmatrix} c \\ d \end{pmatrix}$ is a primitive vector. On the other hand, if $v = (m, n)$ is a primitive vector, then m and n are coprime which implies that there exists $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Now set $g = \begin{pmatrix} b & -a \\ m & n \end{pmatrix}$. This g is clearly in Γ. We conclude the proof with the fact that $v = e_2 g$□

We will now move on to some equidistribution results. For that we however need a general result regarding sequences.

Lemma 3 *Let $\phi : (-\infty, 0) \to \mathbb{C}$ and $a \in \mathbb{C}$. The following are equivalent:*

1. *$\forall \epsilon > 0$ there exists $\Gamma_\epsilon \in (-\infty, 0)$ such that $\forall t \leq \Gamma_\epsilon$ it holds that $|\phi(t) - a| < \epsilon$.*
2. *For every sequence $(t_n)_{n\ in\mathbb{N}}$ of negative numbers for such that $t_n \to -\infty$, it holds that $\phi(t_n) \to a$ as $n \to \infty$.*

Using this Lemma we can now move on to the first equidistribution result.

Proposition 3 *For $t \in \mathbb{R}$ take $k_t \in K$ arbitrary. Then the sets $\Gamma U a_t k_t$ equidistribute in $\Gamma \setminus G$ as $t \to -\infty$ in the following sense:*

$$\int_0^1 f(\Gamma u_s a_{-t_n} k_{t_n})\, ds \to^{t \to -\infty} \int_{\Gamma \setminus G} f(x)\, dx. \tag{5}$$

Proof:
Using Lemma 3 it will be enough to show that for every sequence $(t_n)_{n \in \mathbb{N}}$ and for all $f \in C_c(\Gamma \setminus G)$ it holds that

$$\int_0^1 f(\Gamma u_s a_{-t_n} k_{t_n})\, ds \to \int_{\Gamma \setminus G} f(x)\, dx. \tag{6}$$

Now note that K is compact and thus every sequence contains a subsequence such that $k_{t_n} \to k \in K$ where we maybe need to adjust the indices. For simplicity denote it as k_n and a_n insted of k_{t_n} and a_{-t_n}. Now one findes the following inequality for $\epsilon > 0$:

$$\begin{aligned} |\int_0^1 f(\Gamma u_s a_n k_n)\, ds - \int_{\Gamma \setminus G} f(x)\, dx| \leq |\int_0^1 f(\Gamma u_s a_n k_n)\, ds - \int_0^1 f(\Gamma u_s a_n k)\, ds| \\ + |\int_0^1 k \cdot f(\Gamma u_s a_n)\, ds - \int_{\Gamma \setminus G} k^{-1} \cdot f(x)\, dx|. \end{aligned} \tag{7}$$

No note that f is uniformfly continuous and thus there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ $|f(xk_n) - f(xk)| < \frac{\epsilon}{2}$ $\forall x \in \Gamma/G$. Now $k \cdot f \in C_c(X)$ such that as $\Gamma U a_n$ equidistributes in $\Gamma \setminus G$ we can find some $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ we have

$$\int_0^1 k^{-1} \cdot f(\Gamma u_s a_n)\, ds - \int_{\Gamma \setminus G} k^{-1} \cdot f(x)\, dx|| < \frac{\epsilon}{2}. \tag{8}$$

Now set $N = max(N_1, N_2)$ and now $\forall n \geq N$ it holds that

$$|\int_0^1 f(\Gamma u_s a_n)\, ds - \int_{\Gamma \setminus G} f(x)\, dx| < \epsilon. \tag{9}$$

Now as every sequence contains a subsequence converging to $\int_{\Gamma \setminus G} f(x)\, dx$, the entire sequence converges to said limit. □

Finally, the last proposition expands the equidistribution results from above.

Proposition 4 *For fixes f, the rate of equidistribution is independent of k_t, meaning that $\forall \epsilon > 0$ there exists $T_\epsilon < 0$ with the right properties for all $k_t \in K$*

Proof:

The proof is done in [1] via a simple contradiction.

References

[1] Manuel W. Luethi. *Counting primitive integer vectors using Eskin-Mcmullen.*