Counting Primitive Integer Vectors Using Eskin-McMullen (Part II)

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Preface

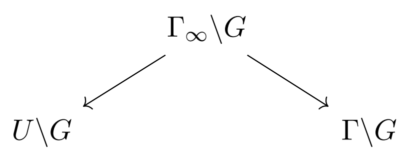
These notes are very heavily based on those from Manuel Luethi from a couple of semesters ago. You can find them at `https://metaphor.ethz.ch/x/2018/fs/401-3370-67L/sc/primitive.pdf`.

Counting

We shall very often make use of the concept of disintegration. If we have a chain of unimodular (sub)groups $H \leq U \leq G$, then integration over $H\backslash G$ can be understood as first going over the quotient $U\backslash G$ and then the fibers $H\backslash U$, that is,

$$\int_{H\backslash G} f(Hg)\,\mathrm{d}m_{H\backslash G}(Hg) = \int_{U\backslash G}\int_{H\backslash U} f(Hug)\,\mathrm{d}m_{H\backslash U}(Hu)\,\mathrm{d}m_{U\backslash G}(Ug)$$

for $f \in C_c(H\backslash G)$. For the following, it will be useful to keep the diagram



of the quotients we can take in mind. The quantity we are interested in - the number of primitive integer vectors in $\mathbb{R}^2 \setminus \{0\} \cong U\backslash G$, is given by the orbit of $U \in U\backslash G$ along the action of Γ (cf. the first half of todays talk).

Proposition 1. *Let $r > 0$ and define the map*

$$F_r : \Gamma\backslash G \to \mathbb{R}, \quad F_r(\Gamma g) := \frac{1}{\mathrm{vol}(B_r(0))}|e_2\Gamma g \cap B_r(0)| = V_r^{-1}|e_2\Gamma g \cap B_r(0)|.$$

Then

$$F_r\,\mathrm{d}m_{\Gamma\backslash G} \to \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\,\mathrm{d}m_{\Gamma\backslash G} = \frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)}\,\mathrm{d}m_{\Gamma\backslash G}$$

for $r \to \infty$ in the weak-star topology. In the above, $m_{\Gamma\backslash G}$ is the finite G-invariant measure on $\Gamma\backslash G$ induced by m_U and $m_{U\backslash G}$.

Note that $F_r(\Gamma)$ gives us exactly what we are looking for, but it is helpful to examine the map as defined above.

Proof. Let $f \in C_c(\Gamma\backslash G)$ be any function. Let $w(\Gamma g) := V_r^{-1}f(\Gamma g)$ be the weight of each $|e_2\Gamma g \cap B_r(0)|$ in the integral. Recall that $e_2\Gamma \cong \Gamma/\Gamma_\infty$ and the isomorphism sends $\Gamma_\infty h \mapsto e_2 h$, therefore, we can make the following decomposition:

$$\int_{\Gamma\backslash G} f \cdot F_r\,\mathrm{d}m_{\Gamma\backslash G} = \int_{\Gamma\backslash G} w(\Gamma g) \sum_{\Gamma_\infty h \in \Gamma_\infty\backslash\Gamma} \mathbf{1}_{B_r(0)}(e_2hg)\,\mathrm{d}m_{\Gamma\backslash G}(\Gamma g)$$

The inner sum is effectively an integral over $\Gamma_\infty\backslash\Gamma$, such that the above formula is a disintegration over $\Gamma_\infty\backslash G$. Since $\Gamma_\infty \leq U \leq G$, we can change the quotient and fiber we integrate over[1], i.e.

$$\begin{aligned}
\int_{\Gamma\backslash G} f \cdot F_r \,\mathrm{d}m_{\Gamma\backslash G} &= \int_{U\backslash G}\int_{\Gamma_\infty\backslash U} w(\Gamma u g)\mathbf{1}_{B_r(0)}(e_2 g)\,\mathrm{d}m_{\Gamma_\infty\backslash U}(\Gamma_\infty u)\,\mathrm{d}m_{U\backslash G}(Ug)\\
&= \int_{U\backslash G}\mathbf{1}_{B_r(0)}(e_2 g)\left(\int_{\Gamma_\infty\backslash U} w(\Gamma u g)\,\mathrm{d}m_{\Gamma_\infty\backslash U}(\Gamma_\infty u)\right)\mathrm{d}m_{U\backslash G}(Ug)\\
&= V_r^{-1}\int_{U\backslash G}\mathbf{1}_{B_r(0)}(e_2 g)\left(\int_0^1 f(\Gamma u_s g)\,\mathrm{d}s\right)\mathrm{d}m_{U\backslash G}(Ug).
\end{aligned}$$

Now take $g \in G$. Using the notation and decomposition from part I, we have $\|e_2 g\| = \|e_2 a(g)\| = \exp(t_g/2)$. Let $\varepsilon > 0$ and choose, by using equidistribution, $T_\varepsilon > 0$ such that

$$\left|\int_0^1 f(\Gamma u_s g)\,\mathrm{d}s - \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G} f\,\mathrm{d}m_{\Gamma\backslash G}\right| < \varepsilon \tag{1}$$

whenever $t_g > T_\varepsilon$. Set $r_\varepsilon = \exp(T_\varepsilon/2)$, and denote

$$K(r_1,r_2) := \{Ug : r_1 < \exp(t_g/2) < r_2\}, \qquad I(g) := V_r^{-1}\int_0^1 f(\Gamma u_s g)\,\mathrm{d}s$$

such that whenever $r > r_\varepsilon$, the previous integral can be written as

$$\int_{K(0,r)} I(g)\,\mathrm{d}m_{U\backslash G}(Ug) = \int_{K(0,r_\varepsilon)} I(g)\,\mathrm{d}m_{U\backslash G}(Ug) + \int_{K(r_\varepsilon,r)} I(g)\,\mathrm{d}m_{U\backslash G}(Ug).$$

By our choice of r_ε and the inequality (1), the second integral satisfies

$$\begin{aligned}
\int_{K(r_\varepsilon,r)} I(g)\,\mathrm{d}m_{U\backslash G}(Ug) &\approx \frac{\mathrm{vol}(K(r_\varepsilon,r))}{\mathrm{vol}(B_r(0))}\left(\frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G} f\,\mathrm{d}m_{\Gamma\backslash G} + \varepsilon\right)\\
&\overset{r\to\infty}{\longrightarrow} \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\left(\int_{\Gamma\backslash G} f\,\mathrm{d}m_{\Gamma\backslash G} + \varepsilon\right).
\end{aligned}$$

On the other hand,

$$\int_{K(0,r_\varepsilon)} I(g)\,\mathrm{d}m_{U\backslash G}(Ug) \to 0$$

for $r \to \infty$, as $V_r^{-1} \to 0$. Since $\varepsilon > 0$ was arbitrary, we are done. □

We are still left with the task to derive a counting statement from the above average. The following lemmas derive the concrete answer to our problem of counting primitive integer vectors.

Lemma 1. *For $r \in \mathbb{R}$, let $\widehat{B}_r := B_{e^r}(0)$. Then*

$$\widehat{F}_r(\Gamma) := F_{e^r}(\Gamma) = \frac{1}{\mathrm{vol}(\widehat{B}_r)}\left|e_2\Gamma \cap \widehat{B}_r\right| \overset{r\to\infty}{\longrightarrow} \frac{1}{\mathrm{vol}(\Gamma\backslash G)}.$$

[1]see the aforementioned notes, especially Corollary 3 for some subtleties we skip over here

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\frac{\operatorname{vol}(\widehat{B}_{r+\delta})}{\operatorname{vol}(\widehat{B}_r)} = \exp(2\delta) < 1 + \varepsilon$$

for all $r \geq 1$. We now construct a symmetric open neighborhood $V \subset G$ of the identity such that $\widehat{B}_r V \subseteq \widehat{B}_{r+\delta}$ holds for all $r \geq 1$. Indeed, recall the KAK-decomposition from part I and set $V := K\{a_t : |t| < 2\delta\}K$. Then, for any $g = ka_tl \in V$ and $v \in \widehat{B}_r$, one has $\|vg\| = \|vka_t\| \leq \exp(|t|/2)\|v\| < \exp(r + \delta)$ by making use of the orthogonality of $k, l \in K$.

Now take any $g \in V$ and calculate

$$\begin{aligned}
\widehat{F}_{r+\delta}(\Gamma g) &= \frac{1}{\operatorname{vol}(\widehat{B}_{r+\delta})}\left|e_2\Gamma g \cap \widehat{B}_{r+\delta}\right| \\
&= \frac{1}{\operatorname{vol}(\widehat{B}_{r+\delta})}\left|e_2\Gamma \cap \widehat{B}_{r+\delta}g^{-1}\right| && g \text{ is bijection on } \mathbb{R}^2 \setminus \{0\} \\
&\geq \frac{1}{\operatorname{vol}(\widehat{B}_{r+\delta})}\left|e_2\Gamma \cap \widehat{B}_r\right| && \text{construction of } V \\
&> \frac{1}{(1+\varepsilon)\operatorname{vol}(\widehat{B}_r)}\left|e_2\Gamma \cap \widehat{B}_r\right| && \text{choice of } \delta \\
&= \frac{1}{1+\varepsilon}\widehat{F}_r(\Gamma).
\end{aligned}$$

If we now have some non-negative $\varphi \in C_c(\Gamma\backslash G)$ with integral 1, support contained in ΓV (which is an open neighborhood of Γ) and which does not vanish at Γ, we observe that

$$\widehat{F}_r(\Gamma) \leq (1+\varepsilon)\int_{\Gamma\backslash G} \widehat{F}_{r+\delta}(x)\varphi(x)\,\mathrm{d}x \overset{\text{Prop. 1}}{\longrightarrow} (1+\varepsilon)\frac{1}{\operatorname{vol}(\Gamma\backslash G)} \qquad \text{for } r \to \infty.$$

On the other hand, we can repeat the same argument as above backwards, obtaining the inequality $\widehat{F}_r(\Gamma) > \widehat{F}_{r-\delta}(\Gamma g)/(1+\varepsilon)$ for all $r > 1 + \delta$, such that

$$\widehat{F}_r(\Gamma) \geq \frac{1}{1+\varepsilon}\int_{\Gamma\backslash G} \widehat{F}_{r+\delta}(x)\varphi(x)\,\mathrm{d}x \overset{\text{Prop. 1}}{\longrightarrow} \frac{1}{1+\varepsilon}\frac{1}{\operatorname{vol}(\Gamma\backslash G)} \qquad \text{for } r \to \infty.$$

By sending $\varepsilon \to 0$, we squeeze the value of $\widehat{F}_r(\Gamma)$. Since the behaviour of $\widehat{F}_r$ is the same as that of F_r as we send $r \to \infty$, the proof is complete. □

The following lemma completes our counting.

Lemma 2. *For our choice of normalization of the Haar measure, we have*

$$\operatorname{vol}(\Gamma\backslash G) = \frac{\pi^2}{3}.$$

Proof. From previous talks, we know that G acts transitively on the upper half plane $\mathbb{H}$, and that for $f \in C_c(G)$,

$$\Lambda_1(f) = \int_{\mathbb{H}}\int_K f(gk)\,\mathrm{d}m_K(k)\,\mathrm{d}m_{\mathbb{H}}(g \cdot i)$$

defines a Haar measure on $G = \mathrm{SL}_2(\mathbb{R})$ (recall that $\mathrm{Stab}_G(i) = K = \mathrm{SO}_2(\mathbb{R})$). Note the use of the hyperbolic area measure $\mathrm{d}m_{\mathbb{H}} = y^{-2}\,\mathrm{d}x\,\mathrm{d}y$. Uniqueness of the Haar measure up to constants implies that there is $C > 0$ such that for all $f \in C_c(G)$ we have

$$\Lambda_1(f) = C\int_{U\backslash G}\int_U f(ug)\,\mathrm{d}m_U(u)\,\mathrm{d}m_{\mathbb{R}^2}(Ug) \tag{2}$$

since the above disintegration also defines a Haar measure on G (recall that $U\backslash G \cong \mathbb{R}^2 \setminus \{0\}$). Now set

$$E = \{z \in \mathbb{H} : |z| \geq 1, \mathrm{Re}(z) \in [-1/2, 1/2]\}$$
$$F = T^1 E \cong \left\{u_s a_t k \in G : s \in [-1/2, 1/2], e^{-t} \geq \sqrt{1-s^2}, k \in K\right\}$$

which is the usual fundamental domain for $\Gamma\backslash G$, at least up to a set of measure zero. Since F is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$, we have

$$\Lambda_1(\mathbf{1}_F) = 2\pi \int_{\mathbb{H}} \mathbf{1}_E \, \mathrm{d}m_{\mathbb{H}} = 2\pi m_{\mathbb{H}}(E)$$

and we calculate

$$m_{\mathbb{H}}(E) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \, \mathrm{d}y \, \mathrm{d}x = 2 \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

resulting in $\Lambda_1(\mathbf{1}_F) = 2\pi^2/3$. It remains to calculate the multiplicative constant C in equation (2). For this, we simply compare the value of Λ_1 and $\Lambda_2 := C^{-1}\Lambda_1$ on the indicator function of

$$A = T^1\{z \in \mathbb{H} : \mathrm{Re}(z) \in [0, 1/2], \mathrm{Im}(z) \geq 1\} = \{u_s a_t k \in G : s \in [0, 1/2], t \leq 0, k \in K\}$$

which is a subset of F and therefore injective. Then

$$\Lambda_1(\mathbf{1}_A) = 2\pi \int_0^{1/2} \int_1^{\infty} \frac{1}{y^2} \, \mathrm{d}y \, \mathrm{d}x = \pi$$
$$\Lambda_2(\mathbf{1}_A) = \frac{1}{2} \mathrm{vol}(B_1(0)) = \frac{\pi}{2}$$

meaning $C = 2$, and thus $\mathrm{vol}(\Gamma\backslash G) = \Lambda_2(\mathbf{1}_F) = \pi^2/3$. □

Finally, our result is

$$\boxed{F_r(\Gamma) \overset{r\to\infty}{\longrightarrow} \frac{\pi^2}{3}}.$$