Haare measure notes

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1 Prerequisites on topology and measure theory

Before looking at the Haare measure we need to review some results and definition from topology and measure theory. An understanding of following is required:

- Topological group
- Local compactness of a topological group
- Hausdorff (T2 space)
- A sigma-Algebra $\mathcal{A}$ of a set X and a measureable space $(X, \mathcal{A}, \mu)$, where μ is a measure.
- The Borell sigma-Algebra of a topological group X denoted by $\mathcal{B}(X)$

Furthermore, the following definition and theorem need to be known as they play an integral part in the theory of Haare measure.

Definition 1 *A radon measure on a topological space X is a measure μ on $(X, \mathcal{B}(X))$ satisfying:*

- $K \subset X$ *compact* $\rightarrow \mu(K) < \infty$,
- $E \subset \mathcal{B}(\mathcal{X}) \rightarrow \mu(E) = inf(\mu(U) : E \subset U, U\, open)$
- E *open in* $X \rightarrow \mu(E) = sup(\mu(k) : K \subset E, K\, compact)$

Theorem 1 *Given a locally compact T2 space X and a positive linear functional* $\lambda : C_c(X) \rightarrow \mathbb{C}$, *then there exists a unique radon measure μ on X such that*

$$\lambda(f) = \int_X f(x)\, \mu(x) \;\; \forall f \in C_c(X) \tag{1}$$

2 Haare measure

Definition 2 *Let G be a locallc compact T2 group. A left/ right Haare Measure on G is a Radon measure μ on $(G, \mathcal{B}(G))$ such that:*

- $U \subset G$ *open and non-empty* $\rightarrow \mu(U) > 0$
- $\forall E \in \mathcal{B}(G)$ *and* $g \in G$ *it holds that* $\mu(gE) = \mu(E)$ *for left Haare measure and* $\mu(Eg) = \mu(E)$ *for right Haare measure.*

The following theorem yields existance and uniqueness results for the Haare measure. Its proof will be omitted.

Theorem 2 *For every locally compact T2 group ther exists a unique (up to scalar multiples) left / right Haare measure.*

To digest this new definition we will look at 2 examples and find that a Haare measure for groups we are familiar with isn't that complicated.

Example 1 *Let G be a discrete topological Group. Clearly G is T2 and locally compact because of discreteness. Furthermore, note that the open sets are the power set of G and thus* $\mathcal{B}(G) = \mathcal{P}(G)$*. Now we claim that the counting measure* $\mu : E \to |E|$ *is a left and right Haare measure.*
Firstly, we need to check that the counting measure is a Radon measure. This is straightforward using the fact that discreteness implies that a set is compact iff it is finite, all measurable sets are finite and some other basic results regarding discrete topology.

Now if U is open and non-emplty, clearly $|U| > 0$ *and also, if E is measurable then* $|E| = |gE| = |Eg| \ \forall g \in \mathcal{B}(G)$.

We conclude that the counting measure is a left and right Haare measure.

Example 2 *Here we consider the topological group* $G = (\mathbb{R}, +)$*. A left / right Haare measure here is just the n-th power of the Lebesque measure as its regular and translation invariant.*

Note that in the above mentioned examples the provided measure is a left and right Haare measure at the same time. To determin weather this is always the case (it is not) we need to introduce some definition and results.

First, we consider how a topological group acts on $C_c(G)$ via the left / right regular representation $\lambda_G(g) f(x) = f(g^{-1}x)$ and $\rho_g(g) f(x) = f(xg)$

Definition 3 *Given a locally compact T2 group G, a left / right Haare functional is a non-trivial positive linear functional on* $C_c(G)$ *which is invariant under* λ_g / ρ_g

Now given the theorem of Riesz, there is a 1 to 1 correspondence between Haare measures and Haare functionals. The following proposition captures said result.

Proposition 1 *Let G be a locally compact T2 group. Then there are the following mutually inverse maps:*

$$\{HaaremeasuresomG\} \iff \{Haarefunctionalson G\} \tag{2}$$

where we go form Haare measures to Haare functionals with integration and vice versa with Riesz.

Now to discuss the difference between the left and right Haare measure, we need to introduce unimodular and the modular function.

Definition 4 *A locally compact T2 space G is unimodular if every left Haare measure is a right Haare measure. Furthermore, given G and a left Haare measure* μ *on G, we call the function* Λ_G *such that* $\mu_g = \Lambda_G(g)\mu$ *where* $\mu_g : E \to \mu(E)$*, the modular function.*

Note that this is well defined as μ_g is also a measure.

Corollary 1 *A locally compact T2 group is unimodular iff* $\Lambda_G = 1$.

Now we might ask ourself if there are any facts about G that immediately imply that it is unimodular. This is infact true. To name a few properties of G that imply unimodular:

- abelian
- compact
- discrete
- topologically simple
- connected semi simple Lie
- connected nilpotens Lie

Furthermore, we can transform any left Haare measure to a right Haare measure by taking $\mu(E)$ and transforming it to $\mu(E^{-1})$

3 Main Example

In this main example we will show that a left Haare measure does not necessarilly need to be a right Haare measure.

Example 3 *Consider the group*

$$P := \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x, y \in \mathbb{R}, x \neq 0\} \tag{3}$$

We claim that $\frac{dxdy}{x^2}$ *is a left Haare measure and that* $dxdy$ *is a left Haare measure. We show this using the relationship between the Haare functional and Haare measure.*
Take $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$ *and* $f \in C_c(P)$ *to show invariance under* λ_g *and* ρ_g*. We calculate:*

$$\int f\left(g^{-1}X\right) \frac{dxdy}{x^2} = \int f\left(\begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\right) \frac{dxdy}{x^2} = \int f\left(\begin{pmatrix} a^{-1}x & a^{-1}y - bx^{-1} \\ 0 & ax^{-1} \end{pmatrix}\right) \frac{dxdy}{x^2}.$$

Now using the substitution $\tilde{x} = a^{-1}x$ *and* $\tilde{y} = a^{-1}y - bx^{-1}$ *and noting that said determinant is equal to* a^2*, we get:*

$$\int f\left(g^{-1}X\right) \frac{dxdy}{x^2} = \int f\left(\begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{x}^{-1} \end{pmatrix}\right) \frac{d\tilde{x}d\tilde{y}}{\tilde{x}^2}.$$

Doing a similar calculation with the right regular representation also yields right invariance.
We can thus conclude that $\frac{dxdy}{x^2}$ *is a left Haare measure and that* $dxdy$ *is a right Haare measure. As these two are not scalar mutliples of each other we conclude that* P *is not unimodular.*