LATTICES IN LINEAR GROUPS

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1. RECAP

Definition 1.1. A *closed linear group* G is any group which can be embedded into

$$\mathrm{GL}_d(\mathbb{R}) = \{v \in \mathrm{Mat}_{d\times d}(\mathbb{R}) \mid v \text{ is invertible}\}$$

for some $d \geqslant 1$ such that the image is a closed set in $\mathrm{GL}_d(\mathbb{R})$.

Example 1.2. A simple example for such a closed linear group would be

$$\mathrm{SL}_d(\mathbb{R}) = \{v \in \mathrm{Mat}_{d\times d}(\mathbb{R}) \mid \det(v) = 1\},$$

since the map $\det\colon \mathrm{GL}_d(\mathbb{R}) \to \mathbb{R}$ is continuous. So $\mathrm{SL}_d(\mathbb{R})$ is the preimage of a closed set, hence closed itself.
Or also for instance $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{I, -I\}$, $\mathrm{SO}(n)$ (rotation group), $\mathrm{U}(n)$ (unitary group) and $\mathrm{SU}(n)$ (special unitary group) are closed linear groups.

The Seminar group from last week already introduced a left-invariant Riemannian metric[1] on any closed linear group G.
Also, they introduced the notion of the Haar measure:

Definition 1.3 (Haar measure)**.** Any metric, σ-compact, locally compact group G has a (left) Haar measure m_G which is characterized (up to proportionality) by the properties

- $m_G(K) < \infty$ for any compact $K \subseteq G$;
- $m_G(O) > 0$ for any non-empty open set $O \subseteq G$;
- $m_G(gB) = m_G(B)$ for any $g \in G$ and measurable $B \subseteq G$.

2. Quotient spaces

Definition 2.1 (Discrete and uniformly discrete)**.** Let (X, d) be a metric space. A subset $D \subseteq X$ is *discrete* if every point $x \in D$ has a neighborhood intersecting D only in the single point x.
Furthermore, a discrete subset D is called *uniformly discrete* if there exists some $\varepsilon > 0$ such that $d(a, b) > \varepsilon$ for all $a, b \in D$ with $a \neq b$. Meaning that there is a fixed minimal distance between all elements of D.

Example 2.2 (Discrete subsets)**.** One of the simplest example will be $\mathbb{Z} \leqslant \mathbb{R}$ or $\mathbb{Z} \leqslant \mathbb{R}^2$ and generally $\mathbb{Z}^n$ is discrete in $\mathbb{R}^n$. Another example will be $\{e^{\frac{2\pi ik}{l}}\} \leqslant S^1$ with $0 \leqslant k \leqslant l-1$. If we look at the matrix space we will find $\mathrm{SL}_2(\mathbb{Z}) \leqslant \mathrm{SL}_2(\mathbb{R})$ and generally $\mathrm{SL}_n(\mathbb{Z}) \leqslant \mathrm{SL}_n(\mathbb{R})$.

Before introducing the metric on the quotient spaces, we will first go over some examples from topology.

Example 2.3 (Quotient spaces)**.** In the following, check out these quotient spaces and their corresponding isomorphisms (see Figures 1, 2 and 3):

[1] A *Riemannian metric* is the collection of all inner products $\langle\cdot,\cdot\rangle_x$ for any point x in a manifold M on a tangent space $\mathrm{T}_x M$.

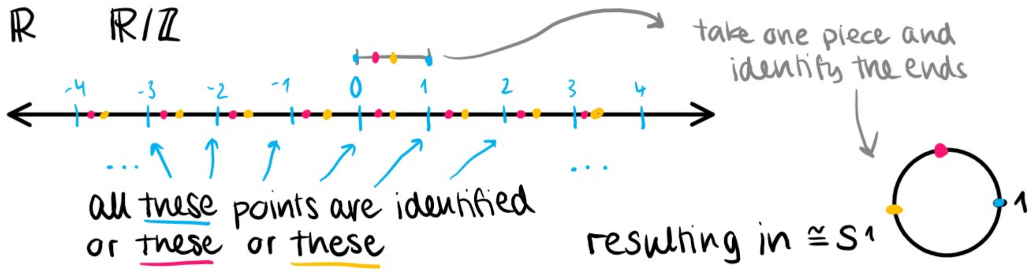

Figure 1. $\mathbb{R}/\mathbb{Z} \cong S^1$.

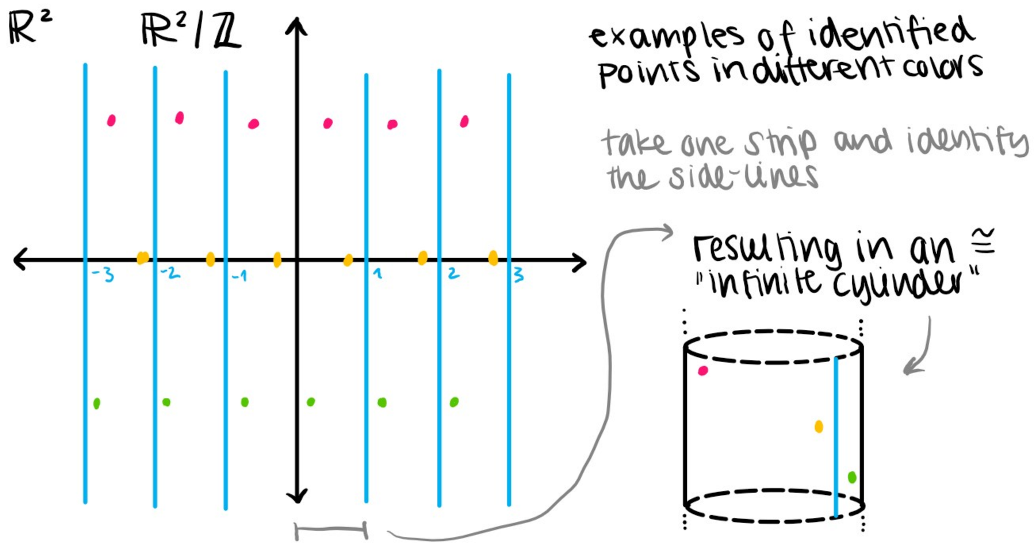

Figure 2. $\mathbb{R}^2/\mathbb{Z} \cong$ "infinite cylinder".

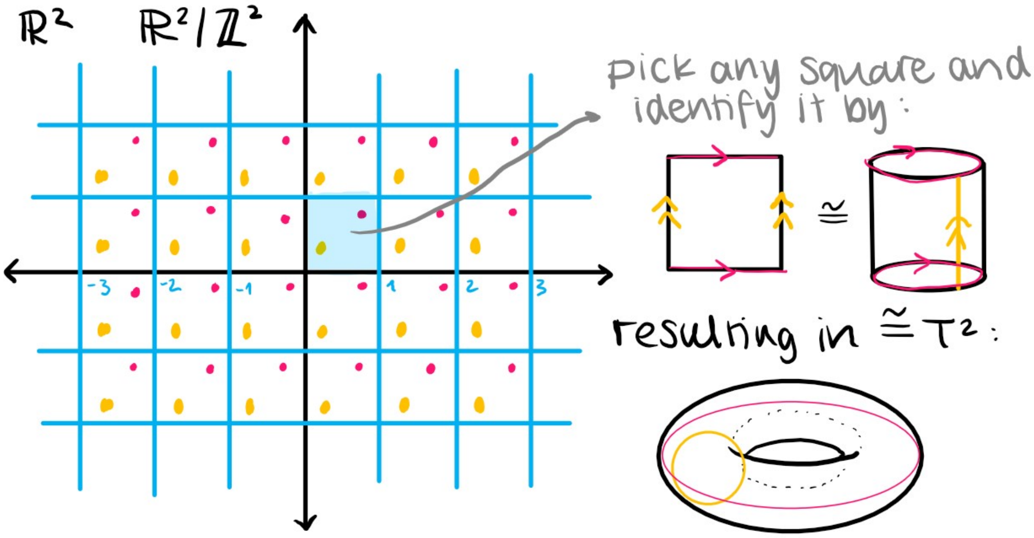

Figure 3. $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$ the two-torus.

From now on, let G be a locally compact σ-compact[2] metric group endowed with a left invariant metric giving rise to the topology of G. Here d_G could be the Riemannian metric on a connected Lie Group G. Since the metric is left invariant, it implies that

$$d_G(g, I) = d_G(g^{-1}g, g^{-1}) = d_G(g^{-1}, I)$$

for any $g \in G$. If Γ is a discrete subgroup , then there is an induced metric on the quotient space $X = \Gamma\backslash G$ defined by

$$d_X(\Gamma g_1, \Gamma g_2) = \inf_{\gamma_1,\gamma_2 \in \Gamma} d_G(\gamma_1 g_1, \gamma_2 g_2) = \inf_{\gamma\in\Gamma} d_G(\gamma g_1, g_2)$$

for any $\Gamma g_1, \Gamma g_2 \in X$. We note that $d_X(\cdot,\cdot)$ indeed defines a metric on X and that we will always use the topology induced by this metric. One of the sequences of the definition of this metric is that X and G are locally isometric in the following sense.

Lemma 2.4 (Injectivity radius)**.** *Let G be a closed linear group equipped with a left-invariant metric and $\Gamma \leqslant G$ be a discrete subgroup. Then for any compact subset $K \subseteq X = \Gamma\backslash G$ there exists some $r = r(K) > 0$, called the* injectivity radius *on K, with the property that for any $x \in K$ the map from*

$$B_r^G = \{g \in G \mid d_G(g, e) < r\}$$

to

$$B_r^X = \{y \in X \mid d_X(x, y) < r\}$$

defined by $g \mapsto xg$ is an isometry. If $K = \{x\}$ where $x = \Gamma h$ for some $h \in G$, then

$$r = \frac{1}{4} \inf_{\gamma\in\Gamma\backslash\{I\}} d_G(h^{-1}\gamma h, I) \tag{1}$$

has this property.

Proof. We first look at the case $K = \{x\}$ where $x = \Gamma h$. Let $r > 0$ be as in equation (1), which is positive since $h^{-1}\Gamma h$ is also a discrete subgroup. Then for g_1, $g_2 \in B_r^G$,

$$d_X(\Gamma h g_1, \Gamma h g_2) = \inf_{\gamma\in\Gamma} d_G(hg_1, \gamma h g_2) = \inf_{\gamma\in\Gamma} d_G(g_1, h^{-1}\gamma h g_2)$$

We wish to show that the infimum is achieved for $\gamma = e$. Suppose that $\gamma \in \Gamma$ has

$$d_G(g_1, h^{-1}\gamma h g_2) \leqslant d_G(g_1, g_2) < 2r$$

then

$$d_G(h^{-1}\gamma h g_2, I) \leqslant d_G(h^{-1}\gamma h g_2, g_1) + d_G(g_1, I) < 2r + r = 3r$$

since $g_1 \in B_r^G$, and similarly

$$d_G(h^{-1}\gamma h, I) = d_G(I, h^{-1}\gamma^{-1}h) \leqslant d_G(I, g_2) + d_G(g_2, h^{-1}\gamma^{-1}h) \leqslant r + d_G(h^{-1}\gamma^{-1}hg_2, I) < 4r$$

Since $g_2 \in B_r^G$ and using left-invariance. Now we can insert equation (1) for r, which implies that $\gamma = I$.

The Proposition now follows by compactness of K. For any x_0 and r as above it is checked that any $y \in B_{r/2}^X(x_0)$ satisfies the first claim of the proposition with r replaced by $r/2$. Hence, K is can be covered by balls so that on each ball there is a uniform injectivity radius. Now according to the compactness of K we can take a finite subcover and the minimum of the associated injectivity radii. □

[2] *σ-compact* means it is the union of a countable set of compact subsets.

Remark 2.5. Notice that given an injectivity radius, any smaller number will also be an injectivity radius. We define the maximal injectivity radius r_x at $x \in X$ as the supremum of the possible injectivity radii for the set $K = \{x\}$ as in Lemma 2.4. If $x = \Gamma h$ then

$$\frac{1}{4} \inf_{\gamma \in \Gamma \setminus \{I\}} d_G(h^{-1}\gamma h, I) \leqslant r_x \leqslant \inf_{\gamma \in \Gamma \setminus \{I\}} d_G(h^{-1}\gamma h, I)$$

by Lemma 2.4.

Example 2.6. A simple example is an injectivity radius for $K = \{x\} \subseteq X = \mathbb{R}^2/\mathbb{Z}^2$ (see Figure 4).

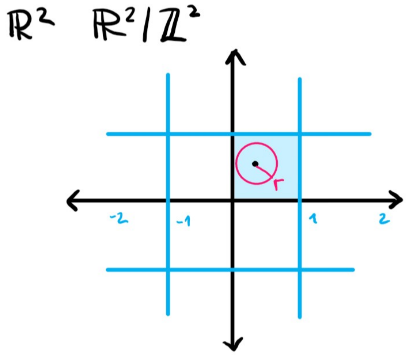

Figure 4. An injectivity radius r for a point $\{x\} \subseteq \mathbb{R}^2/\mathbb{Z}^2$.

We also define the natural quotient map

$$\pi_X : G \longrightarrow X = \Gamma \backslash G$$
$$g \longrightarrow \Gamma g$$

and note that π_X is locally an isometry by left invariance of the metric and Lemma 2.4. One way to understand the quotient space $X = \Gamma \backslash G$ may be to consider a subset $F \subseteq G$ for which the projection π_X, when restricted to F, is a bijection. This motivates the following definition.

Definition 2.7 (Fundamental domain)**.** Let $\Gamma \leqslant G$ be a discrete subgroup. A *strict fundamental domain* $F \subseteq G$ is a measurable set with the property that

$$G = \bigsqcup_{\gamma \in \Gamma} \gamma F$$

Equivalently, $\pi_X|_F : F \to \Gamma \backslash G$ is a bijection.
A measurable set $B \subseteq G$ is called *injective (for* Γ*)* if $\pi_X|_B$ is an injective map, and *surjective (for* Γ*)* if $\pi_X(B) = \Gamma \backslash G$.
However in some cases we could use a slightly more relaxed definition of fundamental domain, which is not strict. This means that the fundamental domain F only needs to fulfill the following two properties: the intersection of F and γF with $\gamma \in \Gamma \setminus \{I\}$ is a null set and $G = \bigcup_{\gamma \in \Gamma} \gamma F$.

Example 2.8. The set $[0,1)^d \subseteq \mathbb{R}^d$ and $[0,1]^d \subseteq \mathbb{R}^d$ are both fundamental domains for the discrete subgroup $\Gamma = \mathbb{Z}^d \leqslant \mathbb{R}^d = G$, but only $[0,1)^d \subseteq \mathbb{R}^d$ is a strict fundamental domain.
Specifically, see Figure 3 where the light-blue shaded region is an example of a fundamental domain of $\mathbb{Z}^2$ in $\mathbb{R}^2$. (It does not need to be this specific square! Any translation of $[0,1)^2$ would also be a fundamental domain).

Lemma 2.9 (Existence of strict fundamental domains)**.** *Let G be a locally compact σ-compact group equipped with a left invariant metric $d_G(\cdot,\cdot)$. If Γ is a discrete subgroup of G and $B_{inj} \subseteq B_{surj} \subseteq G$ are injective (resp. surjective) sets, then there exists a strict fundamental domain F with $B_{inj} \subseteq F \subseteq B_{surj}$. Moreover, $\pi_X|_F : F \to X = \Gamma\backslash G$ is a bi-measurable bijection for any fundamental domain $F \subseteq G$.*

Proof. Notice first that $d_X(\pi_X(g_1), \pi_X(g_2)) \leqslant d_G(g_1, g_2)$ for all $g_1, g_2 \in G$. Therefore, π_X is continuous and hence measurable. Using the assumption that G is σ-compact and lemma 2.4, we can find a sequence of sets (B_n) with $B_n = g_n B^G_{r_n}$ for $n \geqslant 1$ such that $\pi_{X|B_n}$ is an isometry, and $G = \bigcup_{n=1}^{\infty} B_n$. It follows that for any Borel sets $B \subseteq G$ the image $\pi_X(B \cap B_n)$ is measurable for all $n \geqslant 1$, and so $\pi_X(B)$ is measurable. This implies the final claim of the lemma.
Now let $B_{inj} \subseteq B_{surj} \subseteq G$ be as in the lemma. Define inductively the following measurable subsets of G:

$$F_0 = B_{inj}$$

$$F_1 = B_{surj} \cap B_1 \backslash \pi_X^{-1}(\pi_X(F_0))$$

$$F_2 = B_{surj} \cap B_2 \backslash \pi_X^{-1}(\pi_X(F_0 \cup F_1))$$

and so on. Then $F = \bigsqcup_{n=0}^{\infty} F_n$ satisfies all the claims of the lemma. Clearly F is measurable and $B_{inj} \subseteq F \subseteq B_{surj}$. If now $g \in G$ is arbitrary we need to show that $(\Gamma g) \cap F$ consists of a single element. If Γg intersects B_{inj} nontrivially, then the intersection is a singelton by assumption and F_n will be disjoint to Γg for all $n \geqslant 1$ by construction. If Γg intersects B_{inj} trivially, then we choose $n \geqslant 1$ minimal such that Γg intersects $B_{surj} \cap B_n$. By the properties of B_n this intersection is again a singleton, by minimality of n the point in the intersection also belongs to F_n, and Γg will intersect F_k trivially for $k > n$. Hence in all cases we conclude that $(\Gamma g) \cap F$ is a singleton, or equivalently F is a fundamental domain. □

Lemma 2.10 (Independence of choice of fundamental domain)**.** *Let Γ be a discrete subgroup of G. Any two fundamental domains for Γ in G have the same Haar measure. In fact, if B_1, $B_2 \subseteq G$ are injective sets for Γ with $\pi_X(B_1) = \pi_X(B_2)$ then $m_G(B_1) = m_G(B_2)$.*

Alternatively we may phrase this lemma as follows. For any discrete subgroup $\Gamma < G$, the left Haar measure m_G induces a natural measure m_X on $X = \Gamma\backslash G$ such that

$$m_X(B) = m_G(\pi_X^{-1}(B) \cap F)$$

Proof. Suppose B_1 and B_2 are injective sets with

$$\pi_X(B_1) = \pi_X(B_2)$$

Then

$$B_1 = \bigsqcup_{\gamma \in \Gamma} B_1 \cap (\gamma B_2)$$

and

$$\bigsqcup_{\gamma \in \Gamma} \gamma^{-1}(B_1 \cap \gamma B_2) = \bigsqcup_{\gamma \in \Gamma} (\gamma B_1) \cap B_2 = B_2$$

Note that the discrete subgroup $\Gamma < G$ must be countable as G is σ-compact. Therefore, we see that

$$m_G(B_1) = \sum_{\gamma \in \Gamma} m_G(B_1 \cap \gamma B_2) = \sum_{\gamma \in \Gamma} m_G(\gamma^{-1} B_1 \cap B_2) = m_G(B_2)$$

as required. □

Proposition 2.11 (Finite volume quotients). *Let G be a locally compact σ-compact group with a left-invariant metric d_G , and let $\Gamma \leqslant G$ be a discrete subgroup. Then the following properties are equivalent:*

- *$X = \Gamma\backslash G$ supports a G-invariant probability measure, that is a probability measure m_X which satisfies $m_X(g \cdot B) = m_X(B)$ for all measurable $B \subseteq X$ and all g in G.*
- *There is a fundamental domain F for $\Gamma \subseteq G$ with $m_G(F) < \infty$*
- *There is a fundamental domain $F \subseteq G$ which has finite right Haar measure $m_G^{(r)}(F) < \infty$ and $m_G^{(r)}$ is left Γ-invariant.*

If any (and hence all) of these conditions hold, then G is unimodular (that is, the left-invariant Haar measure is also right invariant).

Part of the proof. For the full proof see [2, Chapter 1.1, page 13]. Here we will only prove that (c) is equivalent to (a).
We will start by proving (a) $\Rightarrow$ (c). Suppose therefore that m_X is a probability measure on $X = \Gamma\backslash G$ invariant under the action of G on the right. Then we can define a measure μ on G via the Riesz representation theorem by letting

$$\int f d\mu = \int \sum_{\pi(g)=x} f(g) dm_X(x) \tag{2}$$

for any $f \in C_c(G)$. Here the function defined by the sum

$$F : x = \Gamma g \to \sum_{\gamma\in\Gamma} f(\gamma g),$$

on the right hand side belongs to $C_c(X)$. Indeed the sum vanishes if $x \notin \pi$ (Supp f) and for every given $g \in G$ (and also on any compact neighborhood of g) the sum can be identified with a sum over a finite subset of Γ which implies continuity.
By invariance of μ under the action of G, we see that $\mu = m_F^{(r)}$ is a right Haar measure on G. By the construction above, $m_G^{(r)}$ is left-invariant under Γ. Finally, (2) extends using dominated and monotone convergence to any measurable non-negative function of on G. Applying this to $f = 1_F$ for a fundamental domain $F \subset G$ shows that $m_G^{(r)}(F) = 1$, hence (c).

Now showing the other direction, suppose that (c) holds, and let F be the fundamental domain. We define a measure m_X on X by

$$m_X(B) = \frac{1}{m_G^{(r)}(F)} m_G^{(r)}(F \cap \pi_X^{-1}(B)).$$

By Lemma 2.10, this definition is independent of the particular fundamental domain used. Thus for $g \in G$ and $B \subseteq X$ we have

$$\begin{aligned} m_X(Bg) &= \frac{1}{m_G^{(r)}(F)} m_G^{(r)}(F \cap \pi_X^{-1}(Bg)) \\ &= \frac{1}{m_G^{(r)}(F)} m_G^{(r)}(F \cap \pi_X^{-1}(B)g) \\ &= \frac{1}{m_G^{(r)}(Fg^{-1})} m_G^{(r)}(Fg^{-1} \cap \pi_X^{-1}(B)) = m_X(B), \end{aligned}$$

since $Fg^{-1} \subseteq G$ is also a fundamental domain. This shows (a).

□

Definition 2.12 (Lattice and Uniform lattice)**.** A discrete subgroup $\Gamma \leqslant G$ is called a lattice if $X = \Gamma\backslash G$ supports a G-invariant probability measure. In this case we also say that X has finite volume. In addition, if the quotient space $X = \Gamma\backslash G$ is compact, then we call the discrete subgroup $\Gamma \leqslant G$ a uniform lattice.

Example 2.13. $\mathbb{Z}^2$ in $\mathbb{R}^2$ is a lattice, since the fundamental domain has finite volume. However, $\mathbb{Z}$ in $\mathbb{R}^2$ is not a lattice.

Proposition 2.14 (Haar measure on $X = \Gamma\backslash G$)**.** *Let G and Γ be as in Proposition 2.11, and suppose in addition that G is unimodular. Then the Haar measure m_G on G induces a locally finite G-invariant measure m_X, also called the Haar measure on $X = \Gamma\backslash G$, such that*

$$\int_G f\, dm_G = \int_X \sum_{\gamma\in\Gamma} f(\gamma g)\, dm_X(\Gamma g) \tag{3}$$

for all $f \in L^1_{m_G}(G)$.

Proof. Since we assume that G is unimodular, the argument that (c) implies (a) in the proof of Proposition 2.11 can be used to define the measure m_X. Once again Lemma 2.10 shows that m_X is independent of the choice of fundamental domain $F \subseteq G$ used in the definition, and shows that m_X is G-invatiant. By definition (3) holds for $f = \mathbb{1}_B$ if $B \subseteq F$ or $B \subseteq \gamma F$ for some $\gamma \in \Gamma$. By linearity (3) also holds for any measurable $B \subseteq G$ and hence for any simple function. In particular, the sum on the right hand side of (3) is a measurable function on X (or equivalently on F). The measurability of the sum and the equality of the integrals now extend by monotone convergence to show that (3) holds for any measurable non-negative function. □

3. A long example on the fundamental region

In this chapter we will look at an important example of non-uniform lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$, which is the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I_2, -I_2\}$. Since we cannot draw the matrix space, we need the help of the upper half-plane model of the hyperbolic plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} | y = \mathrm{Im}(z) > 0\}$ and the unit tangent bundle $T^1\mathbb{H} = \{(z,v) \in T\mathbb{H} \mid \|v\|_z = 1\}$ for a simulation. From the second presentation in this seminar we have already seen that $\mathrm{PSL}_2(\mathbb{R}) \cong T^1\mathbb{H} = \mathbb{H} \times S^1$. The idea here is that PSL_2 can be seen as a 4-dimensional object. With the help of the extra condition, that the determinate is equal to 1, it can be reduced to a 3-dimensional object. So a 2-dimensional point in the upper half-plane $\mathbb{H}$ together with a direction represented by vector with length 1 gives in total also 3-dimension. In particular a matrix $m \in \mathrm{PSL}_2(\mathbb{R})$ will be represented by the point $m(i)$ in the upper half-plane model using the möbius transformation.

Here we will only look at a *fundamental domain*, not a strict one, which means: A *fundamental domain* F for $\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ is a measurable subset of $\mathrm{PSL}_2(\mathbb{R})$ with the property that for every $g \in \mathrm{PSL}_2(\mathbb{R})$ we have $|F \cap \Gamma g| = 1$. In addition, if we could find a fundamental domain E for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$, we could easily derive F from it, which is $F = \{g \in \mathrm{PSL}_2(\mathbb{R}) | g(i) \in E\}$. It also follows that if E is not compact, then F is also not compact.

Proposition 3.1. *The set $E = \{z \in \mathbb{H} \mid |z| \geqslant 1, |\mathrm{Re}(z)| \leqslant \frac{1}{2}\}$ illustrated in Figure 5 is a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$ in the following sense:*

$$A(\gamma E \cap E) = 0 \tag{4}$$

for $\gamma \in \mathrm{PSL}_2(\mathbb{Z})\backslash\{I_2\}$, *and*

$$\mathbb{H} = \bigcup_{\gamma\in\mathrm{PSL}_2(\mathbb{Z})} \gamma E. \tag{5}$$

In particular, $\mathrm{PSL}_2(\mathbb{Z})$ *is a lattice in* $\mathrm{PSL}_2(\mathbb{R})$.

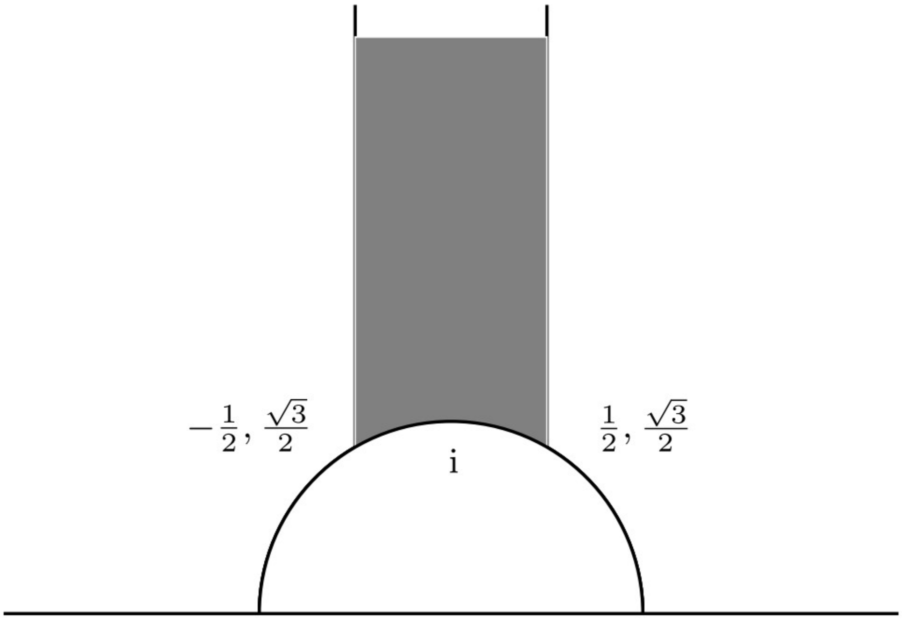

Figure 5. A fundamental region for $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathbb{H}$.

From Figure 5 we can easily see that fundamental domain E is not compact, which implies that fundamental domain F for $\mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ is also not compact and $\mathrm{PSL}_2(\mathbb{Z})$ is not a uniform lattice.

In order to understand the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$, we consider the actions of the elements

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the set $E = \{z \in \mathbb{H} \mid |z| \geqslant 1, |\operatorname{Re}(z)| \leqslant \frac{1}{2}\}$ (as defined in Proposition 3.1).
Now notice that it holds for any $z \in E$ that

$$\sigma z = -\frac{1}{z}, \quad \tau z = z + 1, \quad \text{and} \quad \sigma^2 = (\sigma\tau)^3 = I_2,$$

the identity in $\mathrm{PSL}_2(\mathbb{Z})$.
You can check out the images of E under a few elements of $\mathrm{PSL}_2(\mathbb{Z})$ in Figure 6. We actually only focus on the *boundaries* of E, these are made out of three *geodesics*.[3]
To determine the image of a geodesic (which will also be a geodesic!), it is enough to consider the images of the two limit points of the original geodesic on $\partial\mathbb{H}$.

[3] *Geodesics* are simply the shortest paths between two elements in a given space (e.g. "straight lines" in $\mathbb{R}^2$). In the geometry class of FS23, Mr. Ilmanen called these geodesics for the hyperbolic plane *clines*.

Proof. Let $z \in \mathbb{H}$. We first show that there is some element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $\gamma z \in E$, proving $\mathbb{H} = \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma E$. Recall that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2} \tag{6}$$

Since c and d are integers, there must be a matrix $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with

$$\mathrm{Im}(\gamma z) = \max\{\mathrm{Im}(\eta z) | \eta \in \mathrm{PSL}_2(\mathbb{Z})\}$$

Choose $k \in \mathbb{Z}$ so that $|\mathbb{R}(\tau^k \gamma z)| \leqslant \frac{1}{2}$. We claim that $\omega = \tau^k \gamma z \in E$: if $|\omega| < 1$ then $\mathrm{Im}(-\frac{1}{\omega}) > \mathrm{Im}(z)$, contradicting 4. So $|\omega| \geqslant 1$ and $\omega \in E$ as required.

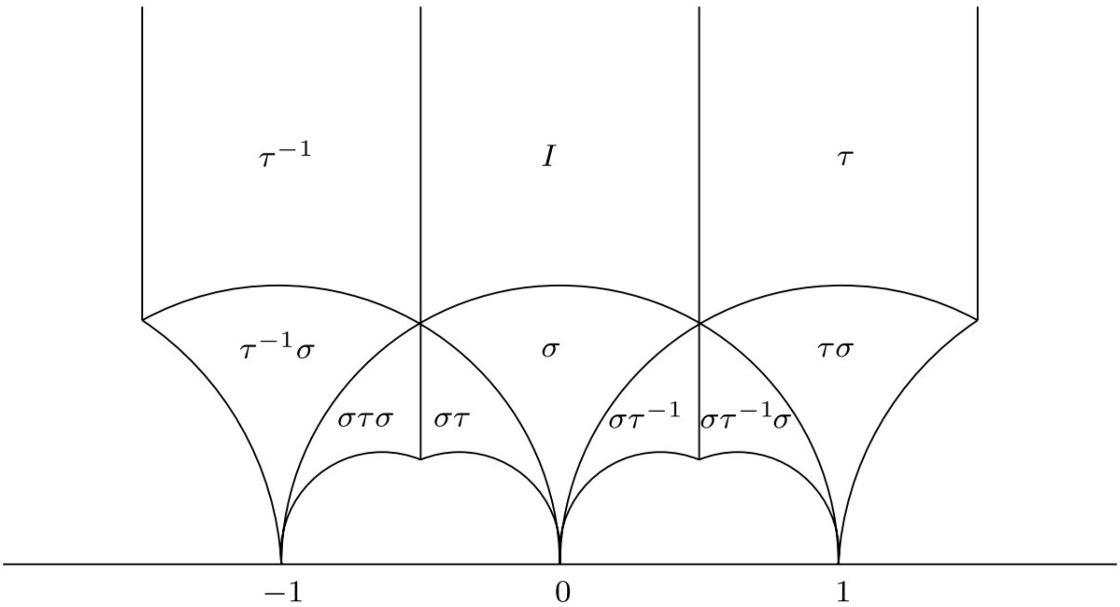

Figure 6. The action of σ and τ on E.

Now let z, $w \in E$ have the property that $\gamma z = \omega$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. We claim that either $|\mathrm{Re}(z)| = \frac{1}{2}$ (and $z = \omega \pm 1$), or $|z| = 1$ (and $\omega = -\frac{1}{z}$). This shows $A(\gamma E \cap E) = 0$. Let γ be given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\mathrm{Im}(\gamma z) < \mathrm{Im}(z)$ replace the pair (z, γ) by $(\gamma(z), \gamma^{-1})$ so that we may assume without loss of generality that $\mathrm{Im}(\gamma z) \geqslant \mathrm{Im}(z)$. This gives $|cz+d| \leqslant 1$ by 6. Since $z \in E$ and $d \in \mathbb{Z}$, this requires that $|c| < 2$, so $c = 0, \pm 1$.
If $c = 0$, then $d = \pm 1$ and the map γ is translation by $\pm b$. By assumption, $|\mathrm{Re}(z)| \leqslant \frac{1}{2}$ and $|\mathrm{Re}(\gamma z)| \leqslant \frac{1}{2}$ so this implies that $b = 0$ and $\gamma = I_2$ or that $b = \pm 1$ and $\{\mathrm{Re}(z), \mathrm{Re}(\gamma z)\} = \{\frac{1}{2}, -\frac{1}{2}\}$.

Now write $\kappa = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $c = 1$, the condition $z \in E$ and $|z+d| \leqslant 1$ implies that $d = 0$ unless $z = -\kappa$ or $z = -\overline{\kappa}$. Taking $d = 0$ forces $|z| \leqslant 1$ and so $|z| = 1$. If $c = -1$ then replace $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, which defines the same element of $\mathrm{PSL}_2(\mathbb{Z})$, and apply the argument above.

This shows that E is a fundamental domain in the sense given.

Finally, to estimate the volume of the fundamental domain E there are two ways. Since we already know from the second presentation, that the ideal triangle in hyperbolic plane has the area π, and the fundamental domain E is contained in an ideal triangle, it follows that volume$(E) \leqslant \pi < \infty$. Or we can also do a calculation. Recall that the hyperbolic area form $dA = \frac{1}{y^2}\,dx\,dy$ on $\mathbb{H}$, and the hyperbolic volume form $dm = \frac{1}{y^2}\,dx\,dy\,d\theta$ on unit tangent bundle $T^1\mathbb{H}$, where θ gives the angle of the unit tangent vector at $z = x + iy$, are both invariant under the respective actions of $\mathrm{PSL}_2(\mathbb{R})$. Notice that any $z \in E$ has $\mathrm{Im}(z) \geqslant \frac{\sqrt{3}}{2}$, so

$$\begin{aligned}\text{volume}(E) &= \int_{z\in E} dA \leqslant \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx, dy}{y^2}\\ &= \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{1}{y^2}dy = \frac{2}{\sqrt{3}} < \infty.\end{aligned}$$

□

4. Divergence of the Quotient by a Lattice

The next proposition explains what it means for a sequence x_n in $\Gamma\backslash G$ to go to infinity (that is, leave any compact subset of X).

Proposition 4.1 (Abstract divergence criterion)**.** *Let G be a locally compact σ-compact group and let $\Gamma < G$ be a lattice. Then the following properties of a sequence (x_n) in $X = \Gamma\backslash G$ are equivalent:*

(i) $x_n \to \infty$ as $n \to \infty$, meaning that for any compact set $K \subseteq X$ there is some $N = N(K) \geqslant 1$ such that $n \geqslant N$ implies that $x_n \notin K$.

(ii) The maximal injectivity radius at $x_n = \Gamma g_n$ goes to zero as $n \to \infty$. That is, there exists a sequence (γ_n) in $\Gamma \smallsetminus \{I\}$ such that $g_n^{-1}\gamma_n g_n \to I \in G$ as $n \to \infty$.

Proof. We note that the two statements in (ii) are equivalent due to the following property

$$\frac{1}{4}\inf_{\gamma\in\Gamma\backslash\{I\}} d_G(h^{-1}\gamma h, I) \leqslant r_{x_0} \leqslant \inf_{\gamma\in\Gamma\backslash\{I\}} d_G(h^{-1}\gamma h, I).$$

Suppose that (i) holds, so that $x_n \to \infty$ as $n \to \infty$. We need to show that the maximal injectivity radius r_{x_n} at x_n goes to zero. So suppose the opposite, then we would have $r_{x_n} \geqslant \varepsilon > 0$ for some $\varepsilon > 0$ and infinitely many n, and by choosing this subsequence we may assume without loss of generality that $r_{x_n} \geqslant \varepsilon > 0$ for all $n \geqslant 1$.
Decreasing ε if necessary, we may assume that $\overline{B_\varepsilon^G}$ is compact (since G is locally compact). Therefore there is some N_1 with

$$x_n \notin x_1\overline{B_\varepsilon^G}$$

for $n \geqslant N_1$. Now remove the terms $x_2, ..., x_{N_1-1}$ from the sequence. Similarly, there is an $N_2 \geqslant 1$ with

$$x_n \notin x_1\overline{B_\varepsilon^G} \cup x_{N_1}\overline{B_\varepsilon^G}$$

for $n \geqslant N_2$. Repeating this process infinitely often, and renaming the thinned-out sequence remaining (x_n) again, we may assume without loss of generality that $d(x_n, x_m) \geqslant \varepsilon$ for all $m \neq n$. This now gives a contradiction to the assumption that X has finite volume: if $x_n = \pi_X(g_n)$ then

$$X \supseteq \bigsqcup_{n=1}^{\infty} x_n B_{\varepsilon/2}^G = \Gamma(\bigsqcup_{n=1}^{\infty} g_n B_{\varepsilon/2}^G),$$

and

$$\bigsqcup_{n=1}^{\infty} g_n B_{\varepsilon/2}^G$$

is a disjoint union of infinite measure, and is an injective set.
Suppose now that (i) does not hold, so there exists some compact $K \subseteq X$ with $x_n \in K$ for

infinitely many n. By Proposition 2.4 there exists an injectivity radius $r > 0$ on K and we see that $r_{x_n} \geqslant r$ for infinitely many n, so that (ii) does not hold either. $\square$

5. Orbits of Subgroups

The next definition will explain what an H-orbit is: You can imagine it in the same way as a normal orbit of an element $x \in X$, however, we only act with H onto X instead of the whole group G. So the H-orbit of an element will be smaller than its "regular" orbit. The same thought also works for the stabilizer subgroup.

Definition 5.1 (H-orbit and stabilizer subgroup)**.** Given an action of G on a space X, which we will write $(x, g) \mapsto g \cdot x$, the H-orbit of $x \in X$ is the set

$$H \cdot x = \{h \cdot x \mid h \in H\} \cong H/\operatorname{Stab}_H(x) \cong \operatorname{Stab}_H(x)\backslash H$$

where

$$\operatorname{Stab}_H(x) = \{h \in H \mid h \cdot x = x\}$$

is the stabilizer subgroup of $x \in X$ and the isomorphisms are sending $h \cdot x$ to $h \operatorname{Stab} H(x)$ resp. to $\operatorname{Stab}_H(x) h^{-1}$. Note that if $X = \Gamma\backslash G$ and $x = \Gamma g$, then

$$\operatorname{Stab}_H(x) = H \cap g^{-1}\Gamma g$$

is a discrete subgroup of H.

Definition 5.2 (Volume of the H-orbit)**.** Fixing a Haar measure m_H on H we define the *volume of the H-orbit*, volume$(H \cdot x)$ to be $m_H(F_H)$ where $F_H \subseteq H$ is a fundamental domain for $\operatorname{Stab}_H(x)$ in H.

Example 5.3 (Orbits of Subgroups)**.** As an example we look at the orbits of the diagonal matrices $H \leqslant \mathrm{GL}_2(\mathbb{R})$ onto $\mathbb{R}^2$. Intuitively, one can say that the smaller the subgroup is, the more orbits there are.

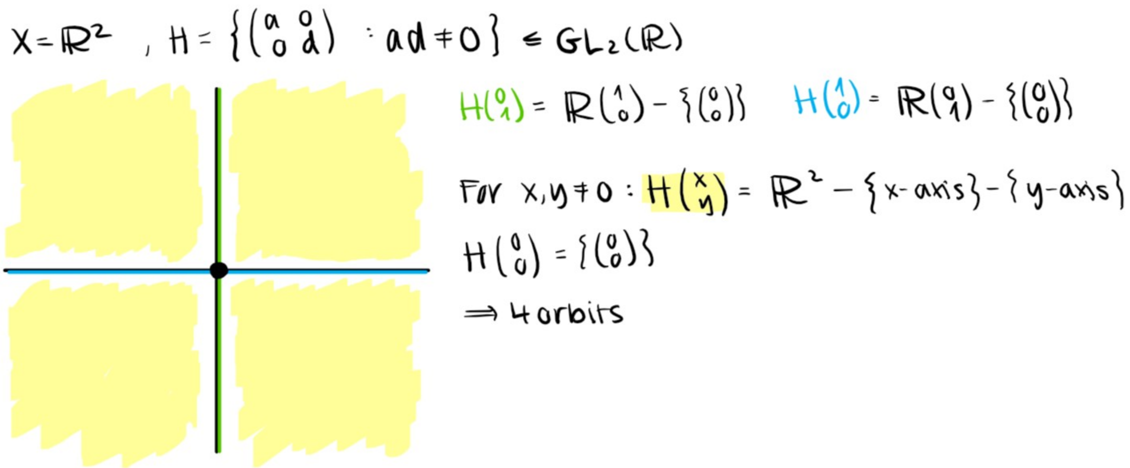

Figure 7. The four orbits generated by the subgroup $H \leqslant \mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$.

Proposition 5.4 (Finite volume orbits are closed)**.** *Let G be a locally compact σ-compact group equipped with a left-invariant metric d, let $\Gamma \leqslant G$ be a discrete subgroup, and let $H \leqslant G$ be a closed subgroup. Suppose that the point $x \in X = \Gamma\backslash G$ has a finite volume H-orbit. Then $xH \subseteq X$ is closed.*

Proof. Suppose that $y \in \overline{xH}$. By Lemma 2.4 there exists a neighborhood B_r^G of $I \in G$ such that the map $g \mapsto yg$ is injective on B_r^G (using the maximal injectivity radius).
Let $V \subseteq H \cap B_{r/2}^G$ be a compact neighborhood of I in H. By assumption, there is a sequence $(z_n)_n$ with $z_n = xh_n = yg_n \in (xH) \cap (yB_r^G)$ for some $h_n \in H, g_n \in B_{r/2}^G$ for each $n \geqslant 1$, and with $g_n \to e$ as $n \to \infty$.

If $z_n V \cap yV \neq \varnothing$ for some n, then $y \in z_n V V^{-1} \subseteq xH$ as desired.
Assume therefore that $z_n V \cap yV = \varnothing$ for all $n \geqslant 1$. Geometrically (and roughly speaking), we may interpret this situation by saying that $z_n V$ approaches yV from a direction transverse to H, see for this Figure 8.

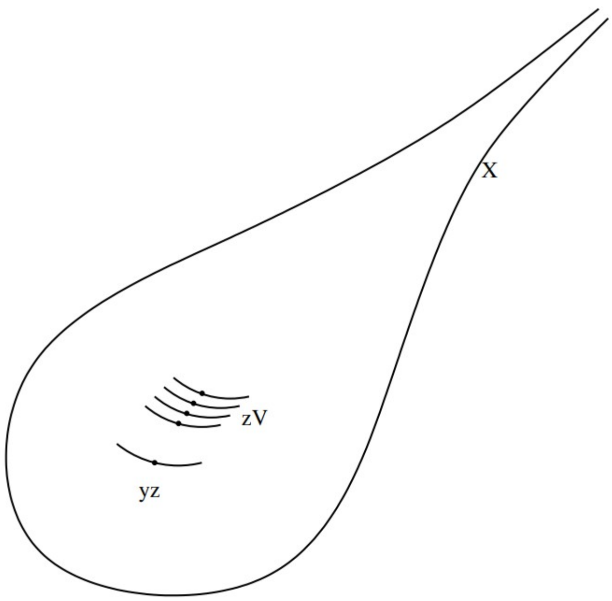

Figure 8. We assume indirectly that the sets $z_n V$ approach yV transverse to the orbit direction.

Now, the compactness of V implies that for any fixed n the set $z_m V$ (which approaches yV) must also be disjoint from $z_n V$ (which has positive distance from yV) for large enough m. Thus we may choose a subsequence and assume that

$$z_n V \cap z_m V = \varnothing$$

for any $n < m$. However, since $z_n = xh_n = yg_n$ as above, each set $z_n V$ is the injective image of the map

$$V \ni h \longmapsto z_n h = y g_n h$$

since $g_n V \subseteq B_{r/2}^G B_{r/2}^G \subseteq B_r^G$. In other words

$$\bigsqcup_{n=1}^{\infty} h_n V \subseteq H$$

is injective for $\mathrm{Stab}_H(x)$. However, this gives

$$m_H\Big(\bigsqcup_{n=1}^{\infty} h_n V\Big) = \infty,$$

which contradicts the assumption that the orbit xH has finite volume.

□

Example 5.5 (Actions on the Two- Torus)**.** In this example we look at the action of the real diagonal matrices H onto $\mathbb{R}^2$ and onto the two-Torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Have a look at Figure 9.

Claim: If for $(x,y) \in T^2$ it holds $\frac{y}{x} \in \mathbb{Q}$, then the H-orbit $H\binom{x}{y}$ is closed. However, for $\frac{y}{x} \in \mathbb{R} - \mathbb{Q}$ irrational, the H-orbit $H\binom{x}{y}$ is not closed.

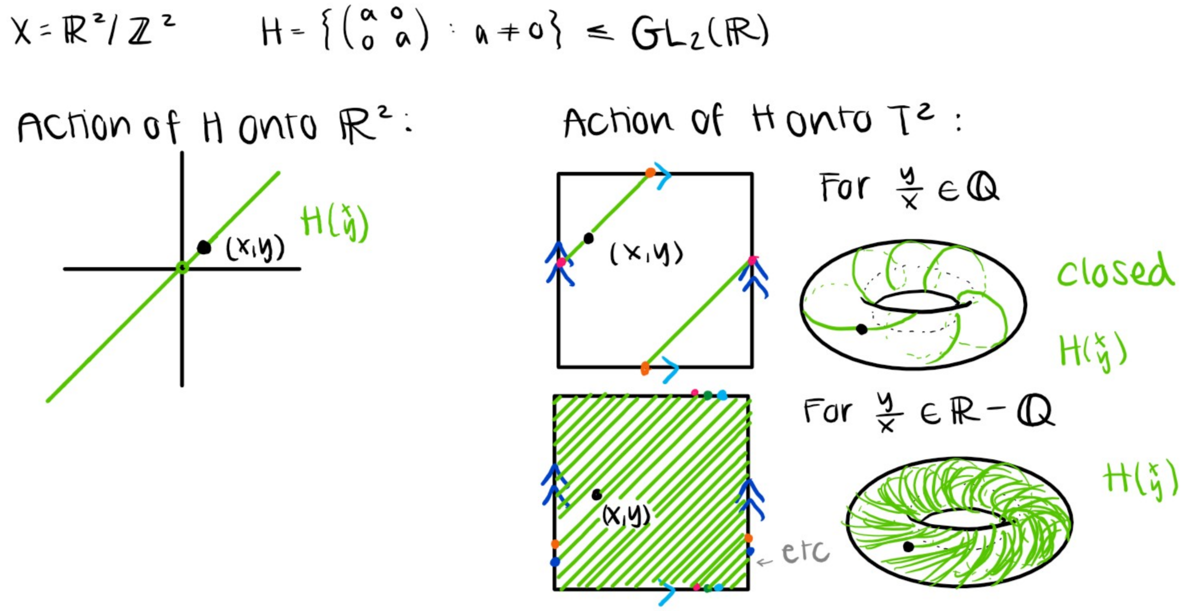

Figure 9. *Left:* Action of H onto $\mathbb{R}^2$. *Right:* Action of H onto T^2 for two different elements of T^2.

Proposition 5.6 (Closed orbits are embedded)**.** *Let G be a locally compact σ-compact group equipped with a left-invariant metric d, let $\Gamma \leqslant G$ be a discrete subgroup, and let $H \leqslant G$ be a closed subgroup. Suppose that the point $x \in X = \Gamma\backslash G$ has a closed H-orbit. Then $xH \subseteq X$ is embedded, meaning that the map $h \in \operatorname{Stab}_H(x)\backslash H \to xh \in xH$ is a homeomorphism. In particular,* volume_{xH} *is a locally finite measure on X.*

6. Dynamics on $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$ and lattices in closed linear groups

In this section we focus more on the intuitive explanations than on rigorous ones. Note that

$$\mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$$

and $\mathrm{SL}_2(\mathbb{R}) = NAK$ with

$$N = \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\right\}, A = \left\{\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a > 0\right\}$$

and $K = \mathrm{SO}(2)$ (which is the stabilizer of $i \in \mathbb{H}$). This can be understood as every $g \in \mathrm{SL}_2(\mathbb{R})$ can be written uniquely as a product $g = nak$ with $n \in N, a \in A$ and $k \in K$.

6.1. **Geodesic Flow.**

The idea of the geodesic flow can be put in simple words: Any point in $\mathbb{H}$ and a direction (indicated with an arrow) will move along some geodesic defined by the *geodesic flow* on $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$:

$$g_t : x \longmapsto x\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} = \begin{pmatrix} e^{-t/2} & \\ & e^{t/2} \end{pmatrix} \cdot x$$

However, if we are in a fundamental domain F, there might be a time where we would leave F. Then there exists a Möbius transformation that will move the point and corresponding arrow back into F. You can imagine this similarly to Example 5.5 with Figure 9, but we move on the geodesics instead of the "straight lines".
Look at Figure 10 where if we reach one of the vertical lines (blue), we simply get horizontally transported to the other vertical line.

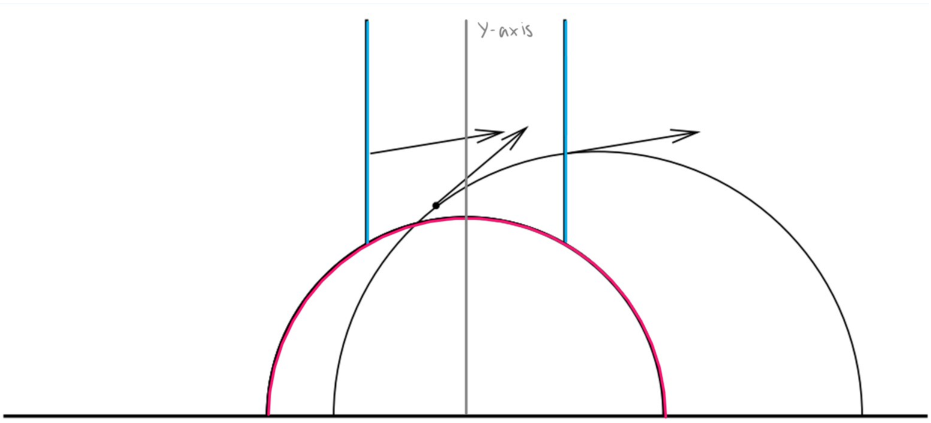

Figure 10. The geodesic flow follows the circle determined by the arrow which intersects $\mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$ normally, and is moved back to F (the fundamental domain) via a Möbius transformation in $\mathrm{SL}_2(\mathbb{Z})$ once the orbit leaves F.

6.2. **Horocyclic Flow.** We recall that the (stable) horocycle flow on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is defined by the action

$$h_s : x \longmapsto x \begin{pmatrix} 1 & -s \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \cdot x$$

for $s \in \mathbb{R}$.
The horocyclic flow works intuitively almost in the same way as the geodesic flow. The main difference is that we do not move along geodesics, but on *horocycles*. Geometrically, the horocycle orbits can be described as circles in the hyperbolic plane, which are tangent to the x-axis or as horizontal lines(see Figure 11).

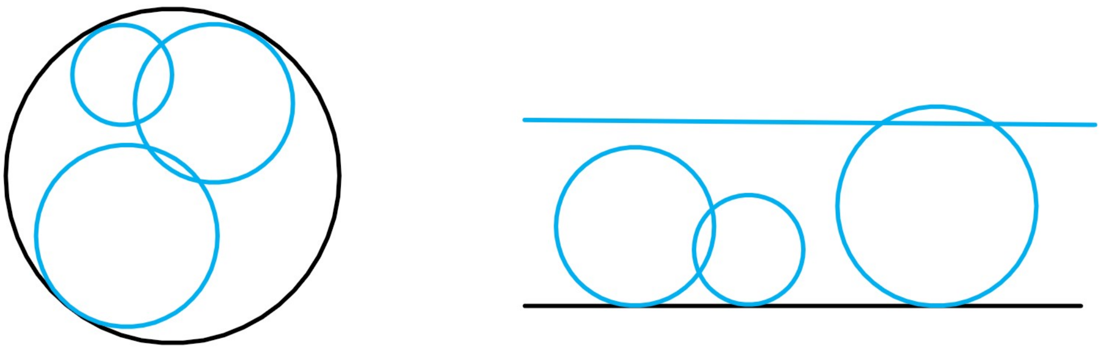

Figure 11. Horocycles in the disk model and in the halfplane model of $\mathbb{H}$.

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