Additional Problems

1. Show that for any root of unity $\zeta \in \mathbb{C}$ whose order is not a prime power, the element $1-\zeta$ is a unit in $\mathcal{O}_{\mathbb{Q}(\zeta)}$.

2. Let K be a number field and let S be a finite set of maximal ideals of $\mathcal{O}_K$. For any $\mathfrak{p} \in S$ and $x \in K^\times$ let $\mathrm{ord}_\mathfrak{p}(x)$ denote the exponent of $\mathfrak{p}$ in the prime factorization of the fractional ideal (x). We define the ring of *S-integers in K* to be
$$\mathcal{O}_{K,S} \;:=\; \bigcap_{\mathfrak{p}\notin S} \mathcal{O}_{K,\mathfrak{p}} \;=\; \big\{x \in K \;\big|\; \forall \mathfrak{p} \notin S : \mathrm{ord}_\mathfrak{p}(x) \geqslant 0\big\}.$$
The group $\mathcal{O}_{K,S}^\times$ is called the group of *S-units in K*.
   (a) Show that the torsion subgroup of $\mathcal{O}_{K,S}^\times$ is $\mu(K)$.
   (b) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the distinct elements of S. Show that the homomorphism
   $$\varphi\colon \mathcal{O}_{K,S}^\times \to \mathbb{Z}^t,\ x \mapsto (\mathrm{ord}_{\mathfrak{p}_i}(x))_i$$
   has kernel $\mathcal{O}_K^\times$ and that its image is a free abelian group of rank t.
   (c) Deduce that $\mathcal{O}_{K,S}^\times \cong \mu(K) \times \mathbb{Z}^{r+s+|S|-1}$.

3. Consider a Dedekind ring A with quotient field K, a finite Galois extension L/K, and let B denote the integral closure of A in L. Consider a subextension K'/K which is also Galois and let A' denote the integral closure of A in K'. Consider a prime $\mathfrak{p}$ of A and a prime $\mathfrak{q} \subset B$ above $\mathfrak{p}$, such that $k(\mathfrak{q})/k(\mathfrak{p})$ is separable. Determine the decomposition of $\mathfrak{p}$ in A' with its numerical invariants r, e, f and its decomposition and inertia groups from the corresponding data in B.

4. Let L/K be a Galois extension of number fields with noncyclic Galois group.
   (a) Show that any prime ideal of $\mathcal{O}_K$ over which lies only one prime ideal of $\mathcal{O}_L$ is ramified in $\mathcal{O}_L$.
   (b) Deduce that there are at most finitely many prime ideals with the property in (a), and in particular no prime ideals of $\mathcal{O}_K$ that are totally inert in $\mathcal{O}_L$.

5. Let K be a quadratic number field and γ the non-trivial Galois automorphism of $K/\mathbb{Q}$. Show that for every fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ the ideal $\mathfrak{a} \cdot \gamma(\mathfrak{a})$ is principal.
   Hint: Prove this first in the case of prime ideals.

6. Consider a prime $p \equiv 3 \bmod (4)$. Show that $K := \mathbb{Q}(\sqrt{-p})$ has odd class number.
   Hint: Use Exercise 5 above and Exercise 5 of Sheet 9.