D-MATH | Number Theory I | HS 2023
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Exercise sheet 1

PRIME IDEALS, INTEGRAL EXTENSIONS, LOCALIZATION, NORMALIZATION

1. Let A be a ring. Prove that a proper ideal $\mathfrak{p} \subsetneqq A$ is a prime ideal if and only if for any ideals $\mathfrak{a}, \mathfrak{b} \subset A$ with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ we have $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

2. Give an example of a ring extension $A \subset B$ and

   (a) prime ideals $\mathfrak{q} \subsetneqq \mathfrak{q}' \subset B$ with $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$.

   (b) a prime ideal $\mathfrak{p} \subset A$ for which there exists no prime ideal $\mathfrak{q} \subset B$ with $\mathfrak{q} \cap A = \mathfrak{p}$.

3. Let $A \subset B$ be an integral ring extension. Show that $a \in A$ is a unit in B if and only if it is a unit in A.

4. Let A be an integral domain and let $S \subset A \setminus \{0\}$ be a multiplicative subset. Prove that the ring extension $A \subset S^{-1}A$ is integral if and only if $S \subset A^\times$.

*5. Let A be an integral domain and let $S \subset A \setminus \{0\}$ be a multiplicative subset. Show that $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ induces a bijection from the set of prime ideals $\mathfrak{q} \subset S^{-1}A$ to the set of prime ideals $\mathfrak{p} \subset A$ satisfying $S \cap \mathfrak{p} = \varnothing$.

   (*Hint:* Show that the inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p} := \left\{\frac{a}{s} \;\middle|\; a \in \mathfrak{p},\ s \in S\right\}$.)

*6. Consider an integral domain A and an element $s \in A \setminus \{0\}$. Show that for the multiplicative subset $S := \{s^n \mid n \geqslant 0\}$ we have

$$S^{-1}A \;\cong\; A[X]/(sX - 1).$$

7. Let $L := k(t)$ be the field of rational functions in one variable over a field k, and let $K := k(s)$ be the subfield generated over k by $s := t + t^{-1}$. Determine the integral closure B of $A := k[s]$ in L.

   (*Hint:* Use Proposition 1.5.2 and compute. Perhaps treat the case $\mathrm{char}(k) = 2$ separately.)