Exercise sheet 10

Different and discriminant

1. Let L/K be a Galois extension of number fields with Galois group Γ, and let $\mathfrak{b}$ be a fractional ideal of $\mathcal{O}_L$. Show that
$$\mathrm{Nm}_{L/K}(\mathfrak{b}) \;=\; K \cap \prod_{\gamma\in\Gamma} {}^{\gamma}\mathfrak{b}.$$

2. Let A be a Dedekind ring with quotient field K. Take finite separable extensions $M/L/K$ and let $C/B/A$ be the respective integral closures of A.
   (a) Prove that $\mathrm{Nm}_{L/K}(\mathrm{Nm}_{M/L}(\mathfrak{c})) = \mathrm{Nm}_{M/K}(\mathfrak{c})$ for any fractional ideal $\mathfrak{c}$ of C.
   (b) Prove that $\mathrm{diff}_{C/A} = \mathrm{diff}_{C/B} \cdot \mathrm{diff}_{B/A}$.

3. For $K := \mathbb{Q}(\sqrt[3]{2})$ compute the prime factorization of the different $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ and verify that a prime ideal of $\mathcal{O}_K$ divides $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ if and only if it is ramified over $\mathbb{Z}$.

4. Let $K := \mathbb{Q}(\alpha)$ for $\alpha := \sqrt[3]{539}$.
   (a) Using Exercise 5 of Sheet 8, show that (7) and (11) are totally ramified in $\mathcal{O}_K$. Let $\mathfrak{p}_7$ and $\mathfrak{p}_{11}$ denote the prime ideals above (7) and (11), respectively.
   (b) Using the discriminant, show that $\mathcal{O}_K = \alpha\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \gamma\mathbb{Z}$, where $\beta := \frac{77}{\alpha}$ and $\gamma := \frac{1+2\alpha+\beta}{3}$, and that $\mathrm{disc}(\mathcal{O}_K) = -3 \cdot 7^2 \cdot 11^2$.
   (c) Show that $3\mathcal{O}_K = \mathfrak{p}_3^2\mathfrak{p}_3'$ for distinct prime ideals $\mathfrak{p}_3$ and $\mathfrak{p}_3'$.
   (d) Show that the different of $\mathcal{O}_K/\mathbb{Z}$ is $\mathfrak{p}_3\mathfrak{p}_7^2\mathfrak{p}_{11}^2$.
   *(e) Using the norm, show that $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ is not principal and conclude that $\mathcal{O}_K$ is not generated by one element over $\mathbb{Z}$.