D-MATH | Number Theory I | HS 2023
Prof. Richard Pink

Exercise sheet 12

Analytic class number formula, density

1. Let Γ be a complete lattice in a finitely dimensional euclidean vector space V of dimension n. Consider a subset $X \subset V$ whose boundary ∂X is $(n-1)$-Lipschitz parametrizable. Show that for $t \to \infty$ we have

$$\left|\Gamma \cap tX\right| \;=\; \frac{\operatorname{vol}(X)}{\operatorname{vol}(V/\Gamma)} \cdot t^n + O(t^{n-1}).$$

   (A subset $Y \subset V$ is k-Lipschitz parametrizable if there exist finitely many Lipschitz continuous maps $[0,1]^k \to Y$ whose images cover Y.)

2. Verify that the analytic class number formula is correct for $K = \mathbb{Q}$.

3. Compute the residue of $\zeta_K(s)$ at $s = 1$ for

   (a) $K = \mathbb{Q}(\sqrt{5})$

   (b) $K = \mathbb{Q}(\sqrt{11})$.

4. For any subset $A \subset \mathbb{Z}_{>0}$ the value

$$\gamma(A) \;:=\; \lim_{x\to\infty} \frac{|\{n \leqslant x : n \in A\}|}{x}$$

   is called the *natural density of* A, if it exists, and the value

$$\mu(A) \;:=\; \lim_{s\to 1+} \frac{\sum_{n\in A} n^{-s}}{\sum_{n\in\mathbb{Z}_{>0}} n^{-s}}$$

   is called the *Dirichlet density of* A, if it exists. Determine both densities of ...

   (a) ... the set of all squares.

   (b) ... the set of positive integers which do not contain the decimal digit 7.

   (c) ... the set of positive integers which have an even number of decimal digits.