Solutions 12

Analytic class number formula, density

1. Let Γ be a complete lattice in a finitely dimensional euclidean vector space V of dimension n. Consider a subset $X \subset V$ whose boundary ∂X is $(n-1)$-Lipschitz parametrizable. Show that for $t \to \infty$ we have

$$\left|\Gamma \cap tX\right| \;=\; \frac{\mathrm{vol}(X)}{\mathrm{vol}(V/\Gamma)} \cdot t^n + O(t^{n-1}).$$

(A subset $Y \subset V$ is k-Lipschitz parametrizable if there exist finitely many Lipschitz continuous maps $[0,1]^k \to Y$ whose images cover Y.)

Solution: See VI §2 Thm. 2 in [Lang: Algebraic Number theory, Springer 1994].

2. Verify that the analytic class number formula is correct for $K = \mathbb{Q}$.

Solution: For $K = \mathbb{Q}$ we have $r = 1$ and $s = 0$, and the class number of $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$ is $h = 1$. As the group of units is finite, the regulator is $R = 1$. Moreover we have $w = |\mathbb{Z}^\times| = 2$ and $d_K = 1$. The analytic class number formula therefore asserts that

$$\mathrm{Res}_{s=1}\,\zeta_\mathbb{Q}(s) \;=\; \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} \;=\; \frac{2^1\cdot(2\pi)^0\cdot 1\cdot 1}{2\cdot 1} \;=\; 1.$$

This agrees with the residue of the Riemann zeta function $\mathrm{Res}_{s=1}\,\zeta(s) = 1$.

3. Compute the residue of $\zeta_K(s)$ at $s = 1$ for

   (a) $K = \mathbb{Q}(\sqrt{5})$

   (b) $K = \mathbb{Q}(\sqrt{11})$.

Solution:

(a) We have $r = 2$ and $s = 0$. By Example 5.1.4, the number of roots of unity of K is $w = 2$. Moreover, by Proposition 3.5.2, the discriminant of K is 5. Using discriminant bounds, we see that every ideal class in $\mathrm{Cl}(\mathcal{O}_K)$ contains an ideal of norm $\leqslant 2$. It can be checked no ideal of norm 2 exists. Hence the class number h is 1.

Next, we compute the regulator of K. In exercise 6 of sheet 7, we have seen that $\frac{1+\sqrt{5}}{2}$ is a fundamental unit of $\mathcal{O}_K^\times$. Moreover, we have $\dim_\mathbb{R}(H) = 1$ by Lemma 5.2.3 and thus $\mathrm{vol}(H/\Gamma) = \log(\frac{1+\sqrt{5}}{2})$.

By Theorem 7.2.7 the residue is therefore

$$\operatorname{Res}_{s=1}\zeta_K(s) = \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} = \frac{2^2\cdot\log(\frac{1+\sqrt{5}}{2})}{2\cdot\sqrt{5}} \approx 0.4304\ldots$$

(b) As in (a), we have $r = 2$ and $s = 0$ and $w = 2$. In exercise 1 of sheet 6, we have seen that the class number h of K is 1.
By Proposition 5.4.2, we have $\mathcal{O}_K^\times = \{a + b\sqrt{11} \mid a, b \in \mathbb{Z},\ a^2 - b^2 11 = \pm 1\}$ and the fundamental unit is the element of $\mathcal{O}_K^\times \cap \mathbb{R}_{>0}$ with the smallest value for b. For $b = 1, 2$, the numbers $b^2 11 \pm 1$ are no square numbers. However, we have $9\cdot 11 + 1 = 100 = 10^2$ and thus $10 + 3\sqrt{11}$ is a fundamental unit of K. As in (a), we get $\operatorname{vol}(H/\Gamma) = \log(10 + 3\sqrt{11})$.
By Theorem 7.2.7 the residue is therefore

$$\operatorname{Res}_{s=1}\zeta_K(s) = \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} = \frac{2^2\cdot\log(10+3\sqrt{11}))}{2\cdot\sqrt{4\cdot 11}} \approx 0.9025\ldots$$

4. For any subset $A \subset \mathbb{Z}_{>0}$ the value

$$\gamma(A) \ := \ \lim_{x\to\infty} \frac{|\{n \leqslant x : n \in A\}|}{x}$$

is called the *natural density of* A, if it exists, and the value

$$\mu(A) \ := \ \lim_{s\to 1+} \frac{\sum_{n\in A} n^{-s}}{\sum_{n\in\mathbb{Z}_{>0}} n^{-s}}$$

is called the *Dirichlet density of* A, if it exists. Determine both densities of ...

(a) ... the set of all squares.

(b) ... the set of positive integers which do not contain the decimal digit 7.

(c) ... the set of positive integers which have an even number of decimal digits.

Solution:

(a) For the set of all squares we have

$$\gamma(A) \ = \ \lim_{x\to\infty} \frac{O(\sqrt{x}\,)}{x} = 0$$

and

$$\mu(A) \ = \ \lim_{s\to 1+} \frac{\sum_{n\geqslant 1} n^{-2s}}{\sum_{n\geqslant 1} n^{-s}} \ = \ \lim_{s\to 1+} \frac{\zeta(2s)}{\zeta(s)} \ = \ \lim_{s\to 1+} \frac{O(1)}{\frac{1}{s-1} + O(1)} \ = \ 0.$$

(b) First we fix an integer $k \geqslant 1$ and let A_k be the set of positive integers n whose last k decimal digits are $\neq 7$. This condition depends only on n modulo 10^k and allows precisely 9^k of the 10^k residue classes modulo 10^k. Among any 10^k successive positive integers there are therefore at most 9^k that lie in A_k. For any $x \in \mathbb{R}$ it follows that

$$N_k(x) \ := \ \big|\{n \leqslant x : n \in A_k\}\big| \ \leqslant \ (\tfrac{9}{10})^k x + 10^k.$$

Therefore

$$\limsup_{x\to\infty} \frac{N_k(x)}{x} \ \leqslant \ \limsup_{x\to\infty} \big((\tfrac{9}{10})^k + \tfrac{10^k}{x}\big) \ = \ (\tfrac{9}{10})^k.$$

On the other hand, for any integer $k \geqslant 1$ there are at most 9^k integers $n \in A \cap [10^{k-1}, 10^k)$. Therefore

$$\sum_{n\in A} n^{-1} \ = \ \sum_{k\geqslant 1} \sum_{\substack{n\in A \\ 10^{k-1}\leqslant n<10^k}} n^{-1} \leqslant \sum_{k\geqslant 1} \frac{9^k}{10^{k-1}} = 90$$

and so

$$\limsup_{s\to 1+} \frac{\sum_{n\in A} n^{-s}}{\sum_{n\geqslant 1} n^{-s}} \ \leqslant \ \lim_{s\to 1+} \frac{90}{O(1) + \frac{1}{s-1}} \ = \ 0.$$

In summary A has natural and Dirichlet density 0.

(c) Let A be the set of positive integers which have an even number of decimal digits. Then for any $k \geqslant 1$, all integers n satisfying $10^{2k-1} \leqslant n < 10^{2k}$ lie in A and all those satisfying $10^{2k} \leqslant n < 10^{2k+1}$ lie outside A. Thus

$$\frac{|\{n \leqslant 10^{2k} : n \in A\}|}{10^{2k}} \ \geqslant \ \frac{|\{10^{2k-1} \leqslant n < 10^{2k}\}|}{10^{2k}} \ = \ \frac{10^{2k} - 10^{2k-1}}{10^{2k}} \ = \ \frac{9}{10}$$

and

$$\frac{|\{n \leqslant 10^{2k+1} : n \in A\}|}{10^{2k+1}} \ \leqslant \ \frac{|\{1 \leqslant n < 10^{2k}\}| + 1}{10^{2k+1}} \ = \ \frac{10^{2k}}{10^{2k+1}} \ = \ \frac{1}{10}.$$

Therefore the natural density of A does not exist.

To compute the Dirichlet density we observe that for any $s > 1$ the function $x \mapsto x^{-s}$ is monotone decreasing; hence for every $n \geqslant 2$ we have

$$\int_n^{n+1} x^{-s}\,dx \ \leqslant \ n^{-s} \ \leqslant \ \int_{n-1}^{n} x^{-s}\,dx.$$

For any $k \geqslant 1$ we therefore have

$$\int_{10^{2k-1}}^{10^{2k}} x^{-s}\,dx \leqslant \sum_{n=10^{2k-1}}^{10^{2k}-1} n^{-s} \ \leqslant \ (10^{2k-1})^{-s} + \sum_{n=10^{2k-1}+1}^{10^{2k}} n^{-s} \ \leqslant (10^{2k-1})^{-s} + \int_{10^{2k-1}}^{10^{2k}} x^{-s}\,dx.$$

Here the integral computes to

$$\int_{10^{2k-1}}^{10^{2k}} x^{-s}\,dx \;=\; \frac{x^{1-s}}{1-s}\bigg|_{10^{2k-1}}^{10^{2k}} \;=\; \frac{(10^{2k})^{1-s}-(10^{2k-1})^{1-s}}{1-s}.$$

Summing over k we deduce that

$$\begin{aligned}
\sum_{n\in A} n^{-s} \;&=\; \sum_{k\geqslant 1}\left(O\big((10^{2k-1})^{-s}\big)+\frac{(10^{2k})^{1-s}-(10^{2k-1})^{1-s}}{1-s}\right)\\
&=\; O\left(\sum_{k\geqslant 1} 10^{-2(k-1)s}\cdot 10^{-s}\right)+\sum_{k\geqslant 1} 10^{2(k-1)(1-s)}\cdot\frac{10^{2(1-s)}-10^{1-s}}{1-s}\\
&=\; O\left(\frac{10^{-s}}{1-10^{-2s}}\right)+\frac{1}{1-10^{2(1-s)}}\cdot\frac{10^{1-s}-10^{2(1-s)}}{s-1}\\
&=\; O(1)+\frac{1}{(10^{s-1}+1)(s-1)}\\
&=\; O(1)+\frac{1}{2(s-1)}
\end{aligned}$$

for $s\to 1+$. Since $\sum_{n\in\mathbb{Z}_{>0}} n^{-s} = \frac{1}{s-1}+O(1)$, we find that the Dirichlet density exists and has the value

$$\mu(A) \;=\; \lim_{s\to 1+}\frac{\frac{1}{2(s-1)}+O(1)}{\frac{1}{s-1}+O(1)} \;=\; \frac{1}{2}.$$