D-MATH | Number Theory I | HS 2023

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Exercise sheet 13

Analytic class number formula, density

1. Determine the Dirichlet density of the set of rational primes $p \equiv 3 \bmod (4)$ that split completely in the field $\mathbb{Q}(\sqrt[3]{2})$.

2. Let L/K be an extension of number fields. Prove that $L = K$ if and only if the set of primes $\mathfrak{p} \subset \mathcal{O}_K$ which are totally split in L has Dirichlet density $> \frac{1}{2}$.

3. Let L/K be an extension of number fields. Prove that L/K is galois if and only if for almost all primes $\mathfrak{p} \subset \mathcal{O}_K$, if there exists a prime $\mathfrak{P}|\mathfrak{p}$ of $\mathcal{O}_L$ with $f_{\mathfrak{P}/\mathfrak{p}} = 1$, then $\mathfrak{p}$ is totally split in $\mathcal{O}_L$.

4. Let a be an integer that is not a third power. Let A be the set of prime numbers p such that $a \bmod (p)$ is a third power in $\mathbb{F}_p$.

   (a) Prove that A and its complement are both infinite.

   (b) Prove that there is no integer N such that the property $p \in A$ depends only on the residue class of p modulo (N).

*5. For $d, N \geqslant 1$, consider the set $P_{d,N}$ of polynomials in one variable of degree at most d whose coefficients have absolute value $\leqslant N$. Consider the subset $Q_{d,N}$ of those polynomials whose Galois group over $\mathbb{Q}$ is the symmetric group S_d. Prove that $\lim_{N\to\infty} \frac{|Q_{d,N}|}{|P_{d,N}|} = 1$.

   Hint: Look at the factorization of polynomials modulo prime numbers.

6. Consider an integer $m \geqslant 1$ and let $L \subset \mathbb{Q}(\mu_m)$ be the intermediate field corresponding to a subgroup $\Gamma < (\mathbb{Z}/m\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$. Express the zeta function $\zeta_L(s)$ as a product of Dirichlet L-functions.