Exercise sheet 2

NOETHERIAN RINGS, DEDEKIND RINGS, LINEARLY DISJOINT EXTENSIONS, FRACTIONAL IDEALS

1. Prove that the following conditions on a ring A are equivalent:

   (a) Every ideal of A is finitely generated.

   (b) Every ascending sequence of ideals of A becomes stationary.

   (c) Every non-empty set of ideals of A possesses a maximal element.

   (a′) Every submodule of a finitely generated A-module is finitely generated.

   (b′) Every ascending sequence of submodules of a finitely generated A-module becomes stationary.

   (c′) Every non-empty set of submodules of a finitely generated A-module possesses a maximal element.

2. Let A be a Noetherian ring. Then for every multiplicative subset $S \subset A$, the ring $S^{-1}A$ is Noetherian.

3. Prove that for any two finite field extensions $L, L'/K$ within a common overfield M the following conditions are equivalent:

   (a) L and L' are linearly disjoint over K, that is, the algebra $L \otimes_K L'$ is a field.

   (b) $[LL'/K] = [L/K] \cdot [L'/K]$

   (c) $[LL'/L] = [L'/K]$

   (d) $[LL'/L'] = [L/K]$

   Moreover, these conditions imply:

   (e) $L \cap L' = K$.

   Conversely, if at least one of L/K and L'/K is galois, then (e) implies the other conditions.

4. Prove that any two finite field extensions $L, L'/K$ with $[L/K]$ and $[L'/K]$ coprime are linearly disjoint over K.

5. Which of the following field extensions are linearly disjoint?

   (a) $\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}$
   (b) $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ and $\mathbb{Q}(i\sqrt[4]{2})/\mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt[5]{2})\mathbb{Q}$ and $\mathbb{Q}(e^{2\pi i/5}\sqrt[5]{2})/\mathbb{Q}$

6. (a) Consider the polynomial ring $A = k[Y, Z]$ over a field k together with the ideal $\mathfrak{a} = (Y, Z)$. Determine the A-submodules

$$\mathfrak{a}^{-1} := \{x \in \mathrm{Quot}(A) \mid x \cdot \mathfrak{a} \subset A\}.$$

   and $\mathfrak{a}\mathfrak{a}^{-1} \subset A$.

   (b) Repeat this for $A = \mathbb{Z}[Y]$ and $\mathfrak{a} = (2, Y)$.

**7. Which of the properties of Dedekind rings hold for the ring $\mathcal{O}(\mathbb{C})$ of entire functions $\mathbb{C} \to \mathbb{C}$?