D-MATH Number Theory I HS 2023
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Exercise sheet 3

Dedekind rings, Fractional ideals, Lattices

1. Show that for all fractional ideals $\mathfrak{a}$ of a Dedekind ring A we have $\mathfrak{a}^{-1}\mathfrak{a} = (1)$.

2. (a) Show that for all fractional ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of a Dedekind ring A we have $(\mathfrak{a}+\mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c}+\mathfrak{b}\mathfrak{c}$ and $(\mathfrak{a}\cap\mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c}\cap\mathfrak{b}\mathfrak{c}$.

   (b) Do the same formulas hold for ideals of an arbitrary ring?

3. Consider non-zero ideals $\mathfrak{a}, \mathfrak{b}$ of a Dedekind ring A with the prime factorizations $\mathfrak{a} = \prod_{i=1}^n \mathfrak{p}_i^{\mu_i}$ and $\mathfrak{b} = \prod_{i=1}^n \mathfrak{p}_i^{\nu_i}$ for distinct maximal ideals $\mathfrak{p}_i$ and exponents $\mu_i, \nu_i \geqslant 0$.

   (a) Prove that
   $$\begin{aligned} \mathfrak{a}+\mathfrak{b} &= \textstyle\prod_{i=1}^n \mathfrak{p}_i^{\min\{\mu_i,\nu_i\}}, \\ \mathfrak{a}\cap\mathfrak{b} &= \textstyle\prod_{i=1}^n \mathfrak{p}_i^{\max\{\mu_i,\nu_i\}}, \\ \mathfrak{a}\cdot\mathfrak{b} &= \textstyle\prod_{i=1}^n \mathfrak{p}_i^{\mu_i+\nu_i}. \end{aligned}$$

   (b) Explain which of these operations can be viewed as the greatest common divisor, respectively the least common multiple, of ideals.

   (c) Deduce Proposition 1.11.5.

4. Prove that a Dedekind ring is factorial if and only if it is a principal ideal domain.

5. Consider the number field $K := \mathbb{Q}(\sqrt{-5})$ and its ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$.

   (a) Show that $(3) = \mathfrak{p}\mathfrak{p}'$ with prime ideals $\mathfrak{p} := (3, 1+\sqrt{-5})$ and $\mathfrak{p}' := (3, 1-\sqrt{-5})$.

   (b) Determine the structure of the ring $\mathcal{O}_K/(3)$.

   (c) Determine the inverse of $\mathfrak{p}$ as a fractional ideal.

   (d) Which powers of the ideal $\mathfrak{p}$ are principal?

   (e) Compute the factorization of (2) into prime ideals.

   (f) Compute the factorization of (5) into prime ideals.

   (g) Compute the factorization of (11) into prime ideals.

6. Show that a subgroup Γ of a finite-dimensional $\mathbb{R}$-vector space V is a complete lattice if and only if Γ is discrete and V/Γ is compact.