Exercise sheet 6

Ideal Class Group

1. (a) Show that the number fields $\mathbb{Q}(\sqrt{11})$ and $\mathbb{Q}(\sqrt{-11})$ have class number 1.
   (b) Show that the class group of $\mathbb{Q}(\sqrt{-14}\,)$ is cyclic of order 4.

2. (a) Let K be a number field. Let $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}_K$ and $m \geqslant 1$ an integer such that $\mathfrak{a}^m = (\alpha)$. Let L/K be a finite extension containing an element $\sqrt[m]{\alpha}$ such that $\sqrt[m]{\alpha}^{\,m} = \alpha$. Show that $\mathfrak{a}\mathcal{O}_L = \sqrt[m]{\alpha}\,\mathcal{O}_L$.
   (b) Deduce that there is a finite field extension L/K such that for every fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ the ideal $\mathfrak{a}\mathcal{O}_L$ is principal.

3. Consider a prime $p \equiv 3 \bmod (4)$. It is known that the class number of $K := \mathbb{Q}(\sqrt{p}\,)$ is odd. Use this fact to prove that there exist $a, b \in \mathbb{Z}$ such that
$$|a^2 - pb^2| = 2.$$
   Hint: Study the ideal $\mathfrak{p} := (2, 1 + \sqrt{p}\,)$.

4. Suppose that the equation $y^2 = x^5 - 2$ has a solution with $x, y \in \mathbb{Z}$.
   (a) Determine the ring of integers and the class number of $K := \mathbb{Q}(\sqrt{-2})$.
   (b) Show that y is odd and that the two ideals $(y \pm \sqrt{-2})$ of $\mathcal{O}_K$ are coprime.
   (c) Prove that $y + \sqrt{-2}$ is a 5-th power in $\mathcal{O}_K$.
   (d) Deduce a contradiction, proving that the equation has no integer solution.

*5. Let $d := -p_1 \cdots p_r$ with distinct primes p_i and $K := \mathbb{Q}(\sqrt{d}\,)$. For any $1 \leqslant i \leqslant r$ consider the ideal $\mathfrak{p}_i := (p_i, \sqrt{d}\,)$ of $\mathcal{O}_K$, and for any subset $I \subset \{1, \ldots, r\}$ consider the ideal $\mathfrak{a}_I := \prod_{i \in I} \mathfrak{p}_i$.
   (a) Show that $\mathfrak{p}_i^2 = (p_i)$.
   (b) Deduce that $\mathfrak{p}_i$ is a maximal ideal above p_i with norm $\mathrm{Nm}(\mathfrak{p}_i) = p_i$.
   (c) Show that $\mathfrak{a}_I$ is principal for $I = \{1, \ldots, r\}$.
   (d) Show that $\mathfrak{a}_I$ is not principal for any $I \neq \varnothing, \{1, \ldots, r\}$.
   (e) Conclude that the class group $\mathrm{Cl}(\mathcal{O}_K)$ contains a subgroup isomorphic to $\mathbb{F}_2^{r-1}$.