Exercise sheet 7

Class number, Discriminant bounds, Units

*1. Let $K := \mathbb{Q}(\sqrt{-\ell}\,)$ for a prime $\ell \equiv 3 \bmod (4)$. Thus complex conjugation is the non-trivial Galois automorphism of $K/\mathbb{Q}$.

(a) Show that every fractional ideal $\mathfrak{b}$ with $\bar{\mathfrak{b}} = \mathfrak{b}$ is principal.

(b) Deduce that for every fractional ideal $\mathfrak{a}$ we have $[\bar{\mathfrak{a}}] = [\mathfrak{a}^{-1}]$ in $\mathrm{Cl}(\mathcal{O}_K)$.

(c) Prove that for any $a \in K^\times$ with $\mathrm{Nm}_{K/\mathbb{Q}}(a) = 1$ there exists $b \in K^\times$ with $a = \bar{b}b^{-1}$. (*Hilbert 90. Hint:* Try $b = \bar{a} + 1$.)

(d) Show that any fractional ideal $\mathfrak{a}$ with $\mathfrak{a}^2$ principal is equivalent to a fractional ideal $\mathfrak{b}$ with $\bar{\mathfrak{b}} = \mathfrak{b}$.

(e) Conclude that the class number of $\mathcal{O}_K$ is odd.

2. Determine all totally real cubic number fields with discriminant ± 4.

Hint: Use a computer algebra system for the actual computation.

3. Work out an analogue of Proposition 5.4.2 in the case $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$.

4. Prove without number theory that the equation $a^2 - b^2 d = -1$ has infinitely many solutions $(a, b) \in \mathbb{Z}^2$ for $d = 2$, but none for $d = 3$. Explain the answer with algebraic number theory.

5. (a) For any number field K, a subring $\mathcal{O} \subset \mathcal{O}_K$ of finite index is called an *order in* $\mathcal{O}_K$. For any such order prove that $\mathcal{O}^\times$ is a subgroup of finite index in $\mathcal{O}_K^\times$.

(b) Consider a squarefree integer $d > 1$ with $d \equiv 1 \bmod (4)$, so that $K := \mathbb{Q}(\sqrt{d}\,)$ has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}\,]$. Explain the precise relation between $\mathbb{Z}[\sqrt{d}\,]^\times$ and $\mathcal{O}_K^\times$.

6. (a) Determine the ring of integers of $K := \mathbb{Q}(\sqrt{5}, i)$.

(b) Determine $\mathcal{O}_F^\times$ for the subfield $F := \mathbb{Q}(\sqrt{5})$.

(c) Find a fundamental unit of $\mathcal{O}_K^\times$.

(d) Show that $|\mu(K)| = 4$ and write down $\mathcal{O}_K^\times$.