Exercise sheet 9

Decomposition of primes

1. Let A be a Dedekind ring with quotient field K. Let K'/K be a finite separable extension and L/K its Galois closure over K. Set $\Gamma := \mathrm{Gal}(L/K)$ and $\Gamma' := \mathrm{Gal}(L/K')$. Let A' be the integral closure of A in K' and B that in L. Consider a maximal ideal $\mathfrak{p} \subset A$ with $k(\mathfrak{p})$ perfect and a prime ideal $\mathfrak{q} \subset B$ above $\mathfrak{p}$.
   (a) Show that $\bigcap_{\gamma\in\Gamma} \gamma^{-1}\Gamma'\gamma = \{1\}$.
   (b) Construct a natural bijection between the set $S_{\mathfrak{p}}$ of prime ideals of A' above $\mathfrak{p}$ and the set of double cosets $\Gamma'\backslash\Gamma/\Gamma_{\mathfrak{q}}$.
   (c) Prove that $\mathfrak{p}$ is totally split in K' if and only if it is totally split in L.
   (d) Prove that $\mathfrak{p}$ is unramified in K' if and only if it is unramified in L.

2. Let A be a Dedekind ring with quotient field K. Consider finite Galois extensions $M/L/K$ such that M/K is Galois. Let $B \subset C$ denote the integral closures of A in $L \subset M$. Consider a prime $\mathfrak{r} \subset C$ above a prime $\mathfrak{q} \subset B$ above a prime $\mathfrak{p} \subset A$.
   (a) Show that the decomposition group of $\mathfrak{r}$ in $\mathrm{Gal}(M/K)$ surjects to the decomposition group of $\mathfrak{q}$ in $\mathrm{Gal}(L/K)$.
   (b) Show that the inertia group of $\mathfrak{r}$ in $\mathrm{Gal}(M/K)$ surjects to the inertia group of $\mathfrak{q}$ in $\mathrm{Gal}(L/K)$, if $k(\mathfrak{p})$ is perfect.
   Hint: Use the multiplicativity $e_{\mathfrak{r}|\mathfrak{p}} = e_{\mathfrak{r}|\mathfrak{q}} \cdot e_{\mathfrak{q}|\mathfrak{p}}$.

3. Construct a number field L in which there are at least two distinct prime ideals of $\mathcal{O}_L$ over every rational prime.
   Hint: Try a composite of quadratic number fields.

4. Consider a number field K and a positive integer m. Let $G_m(K) := \{x^m \mid x \in K^\times\}$ be the subgroup of m-th powers in $K^\times$ and $L_m(K)$ the group of elements $x \in K^\times$ such that, in the prime factorization of (x), all exponents are multiples of m.
   (a) Prove that for every $x \in L_m(K)$, there exists a unique fractional ideal $\mathfrak{a}_x$ such that $(x) = \mathfrak{a}_x^m$.
   (b) Define $S_m(K) := L_m(K)/G_m(K)$ and $\mathrm{Cl}(\mathcal{O}_K)[m] := \{c \in \mathrm{Cl}(\mathcal{O}_K) \mid c^m = 1\}$ and show that we get a well-defined group homomorphism
   $$f\colon S_m(K) \longrightarrow \mathrm{Cl}(\mathcal{O}_K)[m], \quad [x] \mapsto [\mathfrak{a}_x]$$
   (c) Show that f is surjective.
   (d) Identify the kernel of f.

*5. (*Hilbert's Theorem 90*) Let L/K be a finite Galois extension of fields whose Galois group is cyclic and generated by σ. Show that for any element $x \in L^\times$ with $\mathrm{Norm}_{L/K}(x) = 1$ there exists an element $y \in L^\times$ with $x = \sigma(y)/y$.

Hint: Set $n := [L/K]$ and consider the map

$$h\colon L \longrightarrow L, \quad z \mapsto h(z) := \sum_{i=0}^{n-1} \sigma^i(z) \cdot \prod_{i<j<n} \sigma^j(x).$$