1.10 Fractional Ideals

We consider ~~an~~ noetherian integral domain A with quotient field K.

Definition 1.10.1:

(a) A non-zero finitely generated A-submodule of K is called a *fractional ideal of A*.

(b) A fractional ideal of the form $(x) := Ax$ for some $x \in K^\times$ is called *principal*.

(c) The *product* of two fractional ideals $\mathfrak{a}, \mathfrak{b}$ is defined as

$$\mathfrak{a}\mathfrak{b} \;:=\; \left\{ \textstyle\sum_{i=1}^{r} a_i b_i \;\middle|\; r \geqslant 0,\; a_i \in \mathfrak{a}\; b_i \in \mathfrak{b} \right\}.$$

(d) The *inverse* of a fractional ideal $\mathfrak{a}$ is defined as

$$\mathfrak{a}^{-1} \;=\; \left\{ x \in K \;\middle|\; x \cdot \mathfrak{a} \subset A \right\}.$$

Proposition 1.10.2: For any fractional ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we have:

(a) There exist $a, b \in A \setminus \{0\}$ with $(a) \subset \mathfrak{a} \subset (\frac{1}{b})$.

(b) $\mathfrak{a}\mathfrak{b}$ and $\mathfrak{a}^{-1}$ are fractional ideals.

(c) $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ and $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$ and $(1)\mathfrak{a} = \mathfrak{a}$.

(d) $\mathfrak{a} \subset A$ if and only if $A \subset \mathfrak{a}^{-1}$.

Proof (a): Take $x \in \mathfrak{a} \setminus \{0\}$. Write $x = \frac{a}{b}$ with $a, b \in A \setminus \{0\}$
$\Rightarrow bx = a \in \mathfrak{a} \Rightarrow (a) \subset \mathfrak{a}$.
Write $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$ and $x_i = \frac{a_i}{b}$, $a_i \in A$, $b \in A \setminus \{0\}$
$\Rightarrow$ each $x_i \in \langle \frac{1}{b} \rangle \Rightarrow \mathfrak{a} \subset \langle \frac{1}{b} \rangle$.

(b) $\mathfrak{a}\cdot\mathfrak{b}\neq 0$ and $\mathfrak{a}=\langle x_1,\dots,x_n\rangle$, $\mathfrak{b}=\langle y_1,\dots,y_m\rangle \Rightarrow \mathfrak{a}\mathfrak{b}=\langle x_i y_j \mid i,j\rangle$.

$\mathfrak{a}\subset\langle\frac{1}{b}\rangle \Rightarrow b\cdot\mathfrak{a}\subset\langle 1\rangle = A \Rightarrow b\in\mathfrak{a}^{-1}$.

$0\neq\langle a\rangle\subset\mathfrak{a} \Rightarrow \forall x\in\mathfrak{a}^{-1}$: $x\cdot\mathfrak{a}\subset A \Rightarrow x\cdot a\in A \Rightarrow x\in\langle\frac{1}{a}\rangle$

$\Rightarrow \mathfrak{a}^{-1}\subset\langle\frac{1}{a}\rangle$, A noethi $\Rightarrow \mathfrak{a}^{-1}$ fin. gen.

(c) ✓

(d) $\mathfrak{a}\subset A \Leftrightarrow 1\cdot\mathfrak{a}\subset A \Leftrightarrow 1\in\mathfrak{a}^{-1} \Leftrightarrow A\subset\mathfrak{a}^{-1}$.

qed.

Now we assume that A is Dedekind, that is, a noetherien normal integral domain of Krull dimension 1.

Lemma 1.10.3: For every non-zero ideal $\mathfrak{a} \subset A$ there exist an integer $r \geqslant 0$ and maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset \mathfrak{a}$.

Proof: If not, the set $\mathcal{M}$ of counterexamples is nonempty.

A noetherian $\Rightarrow$ $\mathcal{M}$ possesses a maximal element $\mathfrak{a}$.

If $\mathfrak{a} = A$ then $\prod_{i=1}^{0} \mathfrak{p}_i \subset \mathfrak{a}$ $\Rightarrow$ $\lightning$.

If $\mathfrak{a}$ is maximal $\Rightarrow$ $\mathfrak{a} \subset \mathfrak{a}$ $\Rightarrow$ $\lightning$.

So $\mathfrak{a}$ is proper, not maximal, not zero $\Rightarrow$ $\mathfrak{a}$ not prime.

So $\exists\, b_1, b_2 \in A \setminus \mathfrak{a}$ with $b_1 \cdot b_2 \in \mathfrak{a}$.

$\Rightarrow \mathfrak{a} \subsetneqq \mathfrak{a} + (b_1), \mathfrak{a} + (b_2)$

Maximality $\Rightarrow$ $\mathfrak{a} + (b_1), \mathfrak{a} + (b_2) \notin \mathcal{M}$.

So $\exists$ max. ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset \mathfrak{a} + (b_1)$

$\mathfrak{q}_1 \cdots \mathfrak{q}_s \subset \mathfrak{a} + (b_2)$

$\Rightarrow \mathfrak{p}_1 \cdots \mathfrak{p}_r \mathfrak{q}_1 \cdots \mathfrak{q}_s \subset (\mathfrak{a} + (b_1)) \cdot (\mathfrak{a} + (b_2)) \subset \mathfrak{a} \Rightarrow \lightning$. qed.

Lemma 1.10.4: For every maximal ideal $\mathfrak{p} \subset A$ and every fractional ideal $\mathfrak{a}$ we have

(a) $A \subsetneq \mathfrak{p}^{-1}$.

(b) $\mathfrak{a} \subsetneq \mathfrak{p}^{-1}\mathfrak{a}$.

(c) $\mathfrak{p}^{-1}\mathfrak{p} = (1)$.

Proof: (a) $\mathfrak{p} \subset A \Rightarrow A \subset \mathfrak{p}^{-1}$ by 1.10.2. Take any $0 \neq a \in \mathfrak{p}$. By 1.10.3

there exist max. ideals $\mathfrak{p}_i$ with $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset (a)$ and r minimal. $\Rightarrow \mathfrak{p}_1 \cdots \mathfrak{p}_r \subset \mathfrak{p}$.

So $\exists i : \mathfrak{p}_i \subset \mathfrak{p}$. WLOG: $\mathfrak{p}_1 \subset \mathfrak{p}$. Both $\mathfrak{p}_1, \mathfrak{p}$ maximal $\Rightarrow \mathfrak{p}_1 = \mathfrak{p}$.

In particular $r \geq 1$. By minimality: $\mathfrak{p}_2 \cdots \mathfrak{p}_r \not\subset (a)$. Choose $b \in \mathfrak{p}_2 \cdots \mathfrak{p}_r \setminus (a)$.

Then $\frac{b}{a} \notin A$. Also $\mathfrak{p} \cdot b \subset \mathfrak{p} \cdot \mathfrak{p}_2 \cdots \mathfrak{p}_r \subset (a) \Rightarrow \mathfrak{p} \cdot \frac{b}{a} \subset A$. So $\frac{b}{a} \in \mathfrak{p}^{-1}$.

(b) Since $A \subset \mathfrak{p}^{-1}$ we have $\mathfrak{a} \subset \mathfrak{p}^{-1}\mathfrak{a}$. Assume $\mathfrak{a} = \mathfrak{p}^{-1}\mathfrak{a}$. Take $y \in \mathfrak{p}^{-1} \setminus A$ by (a)

$\Rightarrow y \cdot \mathfrak{a} \subset \mathfrak{a}$. Write $\mathfrak{a} = (x_1, \ldots, x_n)$ and $y x_i = \sum_j a_{ij} x_j$ with $a_{ij} \in A$.

Write $M := (a_{ij})$, $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow y \cdot \underline{x} = M \cdot \underline{x}$.

$\Rightarrow (y \cdot I_n - M) \cdot \underline{x} = \underline{0}$. Multiply by the adjoint of $y \cdot I_n - M$

$\Rightarrow \underbrace{\det(y \cdot I_n - M)}_{\in K} \cdot \underbrace{\underline{x}}_{\neq \underline{0}} = \underline{0}$. Since $\underline{x} \neq \underline{0}$ we deduce $\det(y \cdot I_n - M) = 0$.

So $f(y) = 0$ for $f(X) := \det(X \cdot I_n - M) \in A[X]$ monic. $\Rightarrow y$ integral over $A \Rightarrow y \in A$. $\Rightarrow$ contradiction!

(c) $\bar{g}^{-1}g < A$ by the def. of $\bar{g}^{-1}$.
By (b) we have $g \subsetneq \bar{g}^{-1}g$.

$\Rightarrow$ $g \subsetneq \bar{g}^{-1}g < A$.
g max.

$\Rightarrow$ $\bar{g}^{-1}g = A$.

qed.

Theorem 1.10.5: Any non-zero ideal of A is a product of maximal ideals and the factors are unique up to permutation. *(Unique factorization of ideals)*

Proof: Existence: Let $\mathcal{M}$ be the set of counterexamples. Assume $\mathcal{M} \neq \emptyset$.

Let $\mathfrak{a} \in \mathcal{M}$ be a maximal element.

$\mathfrak{a} = A \Rightarrow$ ↯ because $A =$ empty product.

So $\mathfrak{a} \subsetneq A$, so $\exists$ max. ideal $\mathfrak{a} \subset \mathfrak{p} \subset A$. Then $\mathfrak{a} \subsetneq \underbrace{\mathfrak{p}^{-1}\mathfrak{a}}_{\subset A}$.

and $\mathfrak{a} < \mathfrak{p} \Rightarrow \mathfrak{p}^{-1}\mathfrak{a} \subset \mathfrak{p}^{-1}\mathfrak{p} = A$.

Then $\mathfrak{p}^{-1}\mathfrak{a} \notin \mathcal{M}$. So $\mathfrak{p}^{-1}\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ with $\mathfrak{p}_i < A$ maximal

$\Rightarrow \mathfrak{a} = A \cdot \mathfrak{a} = \mathfrak{p}\mathfrak{p}^{-1} \cdot \mathfrak{a} = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{a} = \mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_r \Rightarrow$ ↯.

Uniqueness: Say $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s$

If $r = 0$ then $A = \mathfrak{q}_1 \cdots \mathfrak{q}_s \Rightarrow s = 0$.

Else $r > 0$ and $\mathfrak{q}_1 \cdots \mathfrak{q}_s \subset \mathfrak{p}_1 \Rightarrow \exists i : \mathfrak{q}_i \subset \mathfrak{p}_1 \Rightarrow \mathfrak{q}_i = \mathfrak{p}_1$.

WLOG: $i = 1 \Rightarrow \mathfrak{p}_2 \cdots \mathfrak{p}_r = \mathfrak{p}_1^{-1}\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r = \mathfrak{q}_1^{-1}\mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_2 \cdots \mathfrak{q}_s$.

Finish by induction on r.

q.e.d.