Reminder:

Fix a finite dimensional $\mathbb{R}$-vector space V and an (additive) subgroup $\Gamma \subset V$.

Definition-Proposition 2.1.3: The following are equivalent:

(a) Γ is discrete.

(b) $\Gamma = \bigoplus_{i=1}^{m} \mathbb{Z}v_i$ for $\mathbb{R}$-linearly independent elements $v_1, \ldots, v_m$.

Such a subgroup is called a *lattice*.

Counterexample:

$\mathbb{Z} \oplus \mathbb{Z} \cdot \sqrt{2} \subset \mathbb{R}$.

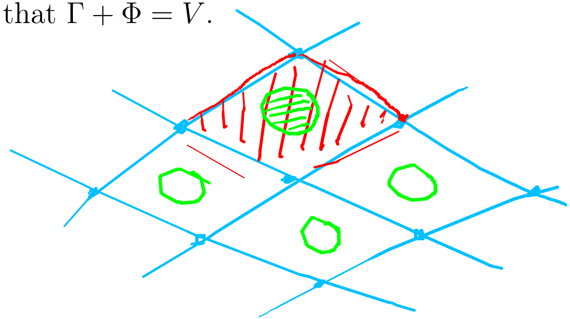
Definition-Proposition 2.1.4: The following are equivalent:

(a) Γ is discrete and there exists a bounded subset $\Phi \subset V$ such that $\Gamma + \Phi = V$.

(b) Γ is discrete and V/Γ is compact.

(c) $\Gamma = \bigoplus_{i=1}^{n} \mathbb{Z}v_i$ for an $\mathbb{R}$-basis $v_1, \ldots, v_n$ of V.

Such a subgroup is called a *complete lattice*.



In the following we consider a lattice $\Gamma \subset V$.

Definition 2.1.5: Any measurable subset $\Phi \subset V$ such that $\Phi \to V/\Gamma$ is bijective is called a *fundamental domain for* Γ.

Example 2.1.6: If $\Gamma = \bigoplus_{i=1}^{n} \mathbb{Z}v_i$ for an $\mathbb{R}$-basis $v_1, \ldots, v_n$ of V, a fundamental domain is:

$$\Phi := \Big\{ \sum_{i=1}^{n} \overset{x_i v_i}{\cancel{x_i}} \;\Big|\; \forall i\colon\ 0 \leqslant x_i < 1 \Big\}.$$

Caution 2.1.7: If $V \neq 0$, there does not exist a compact fundamental domain, because there is a problem with the boundary.

2.2 Volume

$\mathbb{R}^n \xrightarrow{\sim} V$ $\exists$ isometry.

Now we fix a scalar product $\langle\ ,\ \rangle$ on V.

Proposition 2.2.1: (a) There exists a unique Lebesgue measure $d\mathrm{vol}$ on V such that for any measurable function f on V and any orthonormal basis $(e_1, \ldots, e_n)$ of V we have

$$\boxed{\int_V f(v)\ d\mathrm{vol}(v) \;=\; \int_{\mathbb{R}^n} f\big(\textstyle\sum_{i=1}^n x_i e_i\big)\, dx_1 \ldots dx_n.}$$

RHS does not change under an orthogonal matrix.

(b) For any $\mathbb{R}$-basis $(v_1, \ldots, v_n)$ of V we then have

$$\mathrm{vol}\big(\{\textstyle\sum_{i=1}^n x_i v_i \,\big|\, \forall i\colon\ 0 \leqslant x_i < 1\}\big) \;=\; \sqrt{\det\big(\langle v_i, v_j\rangle\big)_{i,j=1}^n}$$

and

$$\int_V f(v)\ d\mathrm{vol}(v) \;=\; \int_{\mathbb{R}^n} f\big(\textstyle\sum_{i=1}^n y_i v_i\big)\, dy_1 \ldots dy_n \cdot \boxed{\sqrt{\det\big(\langle v_i, v_j\rangle\big)_{i,j=1}^n}}.$$

$\mathbb{R}^n \xrightarrow{\sim} V$ isom. of $\mathbb{R}$-vectorspaces.

Definition-Proposition 2.2.2: Consider any fundamental domain $\Phi \subset V$.

(a) For any measurable function f on V/Γ this integral is independent of Φ:

$$\boxed{\int_{V/\Gamma} f(\bar{v})\, d\mathrm{vol}(\bar{v}) \;:=\; \int_{\Phi} f(v+\Gamma)\, d\mathrm{vol}(v).}$$

(b) In particular we obtain

$$\mathrm{vol}(V/\Gamma) \;:=\; \int_{V/\Gamma} 1\, d\mathrm{vol}(\bar{v}) \;=\; \mathrm{vol}(\Phi).$$

$V \to V/\Gamma$

$v \mapsto v+\Gamma$

Φ' another fund. domain.

$\Phi = \bigsqcup_{\gamma\in\Gamma} \Phi \cap (\Phi'+\gamma)$

$\uparrow +\gamma$

$\Phi' = \bigsqcup_{\gamma\in\Gamma} (\Phi-\gamma) \cap \Phi'$

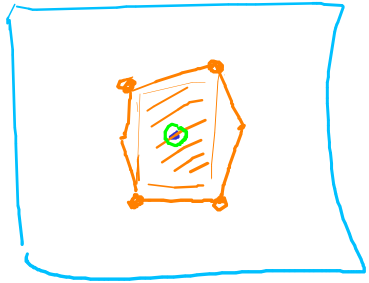
Fact 2.2.3: We have $\mathrm{vol}(V/\Gamma) < \infty$ if and only if Γ is a complete lattice.

2.3 Lattice Point Theorem

Let Γ be a complete lattice in a finite dimensional euclidean vector space V.

Definition 2.3.1: A subset $X \subset V$ is *centrally symmetric* if and only if

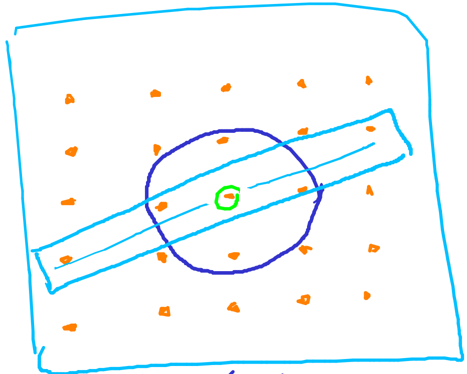
$$X = -X := \{-x \mid x \in X\}.$$



Theorem 2.3.2: Let $X \subset V$ be a centrally symmetric convex subset which satisfies

$$\mathrm{vol}(X) \;>\; 2^{\dim(V)} \cdot \mathrm{vol}(V/\Gamma).$$

Then $X \cap \Gamma$ contains a non-zero element.



Proof: Set $X' := \{ \frac{x}{2} \mid x \in X\}$. Then $\mathrm{vol}(X') > \mathrm{vol}(V/\Gamma)$.

Let Φ be a fund. domain. $\Rightarrow X' = \bigcup_{\gamma \in \Gamma} X' \cap (\Phi + \gamma)$

$\mathrm{vol}(X') = \sum_\gamma \mathrm{vol}(X' \cap (\Phi + \gamma)) = \sum_\gamma \mathrm{vol}((X' - \gamma) \cap \Phi)$

If the $(X'-\gamma) \cap \Phi$ are pair-wise disjoint, then we get $\leq \mathrm{vol}(\Phi) = \mathrm{vol}(V/\Gamma)$.

So they are not, i.e. $\exists\, \gamma, \gamma' \in \Gamma$ distinct: $(X'-\gamma) \cap \Phi \cap (X'-\gamma') \neq \emptyset$

$\Rightarrow \exists\, x, x' \in X'$: $x - \gamma = x' - \gamma'$

$\Rightarrow \Gamma \setminus \{0\} \ni \gamma - \gamma' = x - x' \in X' + X' = X$

$2x, 2x' \in X \Rightarrow -2x' \in X$

$\Rightarrow x - x' = \frac{2x + (-2x')}{2} \in X$ qed.

Remark 2.3.3: The theorem is sharp. For example if $V = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ and $X =]-1, 1[^n$, then we have $\mathrm{vol}(X) = 2^{\dim(V)} \cdot \mathrm{vol}(V/\Gamma)$ and $X \cap \Gamma = \{0\}$.

Application 2.3.4: An n-dimensional ball B_r of radius r has volume in $\mathbb{R}^n$

$$\boxed{\mathrm{vol}(B_r) \;=\; \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot r^n.}$$

Therefore the smallest non-zero vector in Γ has length

$$\boxed{\leqslant \frac{2}{\sqrt{\pi}} \cdot \sqrt[n]{\mathrm{vol}(V/\Gamma) \cdot \Gamma(\tfrac{n}{2}+1)}.}$$

More generally, for every k one can bound the combined lengths of k linearly independent vectors in Γ using *successive minima*.

Proof: $\mathrm{vol}(B_r) > 2^n \cdot \mathrm{vol}(V/\Gamma)$

$$\iff \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \, r^n > 2^n \cdot \mathrm{vol}(V/\Gamma)$$

$$\iff \left(\frac{\sqrt{\pi} \cdot r}{2}\right)^n > \mathrm{vol}(V/\Gamma) \cdot \Gamma(\tfrac{n}{2}+1)$$

$$\iff r > \frac{2}{\sqrt{\pi}} \cdot \sqrt[n]{\mathrm{vol}(V/\Gamma) \cdot \Gamma(\tfrac{n}{2}+1)}$$

qed.